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EFFICIENT PRECONDITIONING FOR THE p-VERSION FINITE ELEMENT METHOD IN TWO DIMENSIONS

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Abstract. We formulate and analyze parallel preconditioners for systems of equations arising from the p-version finite element method. Using new theoretical results for polynomial spaces, we prove that condition number grows as $\log^2 p$, where p is the degree of the polynomial space. Numerical results are presented showing that the condition number indeed grows very slowly with p.

Key words. p-Version Finite Element Method, Preconditioning, Domain Decomposition, Parallel Computation, Polynomial Sobolev Inequality, Polynomial Extension Theorems

1. Introduction. In this paper, we study fast parallel preconditioners for systems of equations arising from the p-version finite element method. The p-version finite element method [4, 5] achieves increase of precision by increasing the degree of elements rather than decreasing their size as the h-version.

The finite element method is based on a variational formulation of the original problem: Find u such that

$$u \in H: \quad a(u,v) = f(v) \; \forall v \in H \tag{1.1}$$

where H is a Hilbert space, $a(u,v) = a(v,u)$ is a bilinear form defined on $H \times H$ and f is a bounded linear functional defined over H. We assume that $a(u,v)$ satisfies

$$C^{-1}\|u\|_H^2 \le a(u,u) \le C\|u\|_H^2, \; C > 0. \tag{1.2}$$

The finite element method consists of choosing a finite dimensional subspace $S \subset H$ and posing problem (1.1) on $S \times S$. In what follows the bilinear form $a(u,v)$ is understood to be on $S \times S$. Selecting basis (shape) functions for S transforms problem (1.1) into the problem of finding the solution of a system of linear equations

$$Ax = y \tag{1.3}$$

where A is a positive definite symmetric matrix.

Our basic approach to solving (1.3) is the preconditioned conjugate gradient method. We construct a preconditioning form $c(u,v)$ such that

$$m_1 c(u,u) \le a(u,u) \le m_2 c(u,u), \quad 0 < m_1 < m_2, \tag{1.4}$$

holds for any $u \in S$. The form $c(u,v)$ is also chosen so that the problem $c(u,v) = g(v)$ (for an appropriate linear form $g(v)$ and a chosen set of basis functions) is easier to solve than the original problem. We show that the relative condition number m_2/m_1 grows at most as fast as $\log^2(p)$ for one type of finite element space, and as $p\log^2 p$ for another.

The solution of the problem with the bilinear form $c(u,v)$ decomposes into local highly parallelizable computations and the solution of a relatively small global auxiliary problem.

We study two different methods. In the first (Section 3), the global problem is identical to the system for $p = 1$, which presents a very small part of the computational cost for high p. This method is related to the domain decomposition method by Bramble, Pasciak, and Schatz [7] for the h-version. For other pertinent considerations, see Babuška, Griebel, and

Pitkäranta [3], Babuška and Elman [1], Babuška, Elman, and Markley [2], and Williams [23]; for numerically computed condition numbers and relation to other methods, see Mandel [16]. For another related method using preconditioning by elements of order higher than one for three-dimensional elasticity, see Mandel [15].

In the second method (Section 5), the global system has one variable per element, which correponds to an avarage or the solution on every element. This method was inspired by the methods of Dryja [11] and Bramble, Pasciak, and Schatz [7] for the h-version in three dimensions, and an analogous method was developed by Mandel [14, 16, 17] for the p-version in three dimensional elasticity. Note that the first method leads to a fast growth of the condition number in the three-dimensional case [16]. The analysis in this paper could be used also for three-dimensional problems subject to availability of appropriate extension theorems.

The paper is organized as follows: For the reasons of clarity of presentation, the methods, their analysis, and practical results are presented in the first five sections of the paper, while technical auxiliary results are given only at the end. In Section 2, we introduce some notation and conventions. In Section 3, we formulate and analyze the first preconditioner. Numerical results for a parallel implementation of this preconditioner are presented in Section 4. Section 5 contains analysis of the second preconditioner. Sections 6 and 7 contain auxiliary results about Sobolev norm estimates for polynomial spaces, which are of separate interest. In Section 6, we prove a discrete Sobolev inequality for polynomials on a segment, and bound the $H_{00}^{1/2}$ norm of a polynomial with zero boundary values in terms of its $H^{1/2}$ norm. Section 7 contains various results about H^1 bounded polynomial extensions of functions defined by polynomials on the the boundary of a triangle or a square. The theoretical results of the last two sections are related to the results of Bramble, Pasciak and Schatz [8, 10, 9, 7] and Widlund [21, 22] for the h-version.

We would like to express our appreciation of the interest and comments of Professor Olof Widlund relating to the results of this paper.

2. Notation, Conventions, and Preliminaries. Let $\Re^2$ be two dimensional Euclidean space and

$$\hat{Q} = \left\{(\xi,\eta) \in \Re^2 \mid |\xi| < 1, |\eta| < 1\right\},$$
$$\hat{T} = \left\{(\xi,\eta) \in \Re^2 \mid 0 < \eta < \sqrt{3}(\xi+1), -1 < \xi < 0, \text{ or } 0 < \eta < \sqrt{3}(1-\xi), 0 < \xi < 1\right\}$$

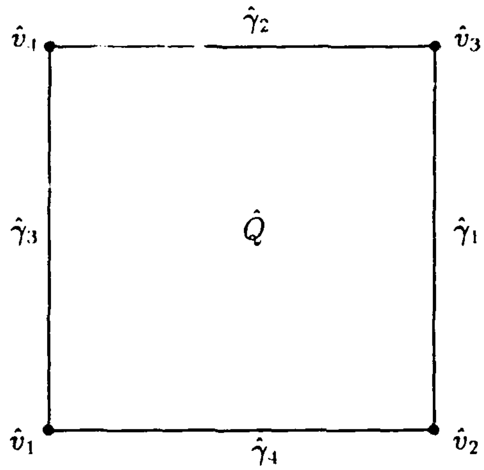
be the reference square and triangle as shown in figures 2.1 and 2.2, respectively. We shall use the generic notation $\hat{K}$ for both $\hat{Q}$ and $\hat{T}$ when the distinction is unimportant.

The image of $\hat{Q}$ (resp. $\hat{T}$) under the mapping F_Q (resp. F_T), $F_Q : \hat{Q} \to Q = F_Q(\hat{Q})$ (resp. $F_T : \hat{T} \to T = F_T(\hat{T})$) is denoted by Q (resp. T). Similarly to before we use the notation K for both Q and T when the distinction is unimportant.

We shall assume that the mapping F_Q is a bijection and that

$$|F_Q|_{1,\infty,\hat{Q}} \le C_1 h_Q, \quad |F_Q^{-1}|_{1,\infty,Q} \le C_2 h_Q^{-1}, \qquad (2.1)$$
$$|J_{F_Q}|_{0,\infty,\hat{Q}} \le C_3 h_Q^2, \quad |J_{F_Q^{-1}}|_{0,\infty,Q} \le C_4 h_Q^{-2}$$

FIG. 2.1. *Reference square*



where J_{F_Q} is the Jacobian of F_Q, $J_{F_Q^{-1}}$ is the Jacobian of F_Q^{-1}, and

$$|F_Q|_{k,\infty,\hat{Q}} = \sup_{\substack{\xi,\eta\in Q\\ |l|=k}} \|D^l F_Q(\xi,\eta)\|_{L^2(\Re^2,\Re^2)}.$$

Similar assumptions are made about F_T. We do not need to assume anything about h_Q or h_T, they are simply numbers proportional to the diameter of K.

Let $\Omega \subset \Re^2$ be a curvilinear polygon, that is, a domain which is bounded by a simple curve consisting of a finite number of smooth arcs with the end points at the vertices of Ω. Further, let $\mathcal{K}$ be a decomposition of Ω into a finite number of curvilinear quadrilaterals or triangles such that

a) $\bar{\Omega} = \bigcup_{K\in\mathcal{K}} \bar{K}$, and for all decompositions under consideration the constants C_i in (2.1) are the same, that is the mappings are uniformly bounded.

b) The intersection $\bar{K}_i \bigcap \bar{K}_j$, $i \neq j$ is either a common vertex or common side, or the intersection is empty.

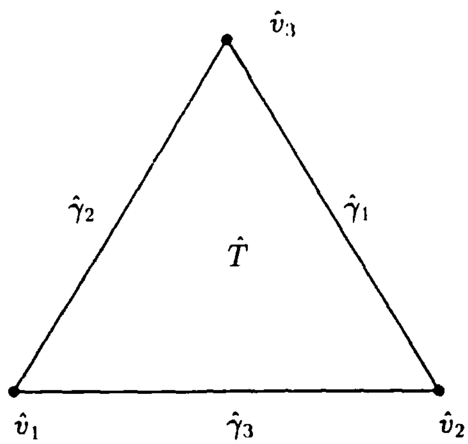
c) If $\bar{K}_i \bigcap \bar{K}_j = \gamma_{i,j}$, $\gamma_{i,j}$ being the common side, then the mappings $F_{K_i}^{-1}$ and $F_{K_j}^{-1}$ coincide on $\gamma_{i,j}$ in the usual sense of the finite element method.

d) The vertices of Ω coincide with the vertices of some K.

In this paper C denotes a generic constant which does not depend on p or any of the functions involved, but which may take different values in different places, even in the same formula.

Let us now define the finite element spaces on $\hat{S}$ and $\hat{T}$. By $\mathcal{P}_p^2(\hat{Q})$, $p \geq 1$, we denote the set of all polynomials on $\hat{Q}$ which are of degree at most p separately in the variables ξ

FIG. 2.2. *Reference triangle*



and η. This space is usually called the space of tensor product polynomials. By $\mathcal{P}_p^3(\hat{Q})$, we denote the span of the union of set of all polynomials of (total) degree at most $\leq p$ with the polynomials which are of degree at most p in ξ, degree 1 in η and of degree 1 in ξ, degree at most p in η. This space coincides with the space of the *serendipity element* [24]. The space $\mathcal{P}_p^3(Q)$ is the minimal space which includes all polynomials of (total) degree p and has basis functions which can be factorized into nodal, side and internal shape functions. This guarantees good approximation properties. In [3], the set $\mathcal{P}_p^2(\hat{Q})$ is denoted by Q_p and the set $\mathcal{P}_p^3(\hat{Q})$ by Q'_p. By $\mathcal{P}_p^1(\hat{T})$ we denote the set of all polynomials of (total) degree at most p, and, for an interval I, $\mathcal{P}_p(I)$ is the set of all polynomials of degree at most p on I. The generic notation $V_p(\hat{K})$ will be used for $\mathcal{P}_p^i(\hat{K})$.

We shall assume that the basis (shape) functions of $V_p(\hat{K})$ can be divided into the following three sets:

a) The set $\mathcal{I}$ of *internal* shape functions. These shape functions are zero on $\partial\hat{K}$.

b) The sets Γ_i, $i = 1, \ldots, n$ ($n = 3$ for $\hat{K} = \hat{T}$ and $n = 4$ for $\hat{K} = \hat{Q}$) of *side* shape functions. If $\hat{\gamma}_i$ is a side of $\hat{K}$ then a side shape function associated with $\hat{\gamma}_i$ is zero on $\partial\hat{K} \setminus \hat{\gamma}_i$. Γ_i is the set of all side shape functions associated with $\hat{\gamma}_i$.

c) The sets $\mathcal{N}_i$, $i = 1, \ldots, n$ of the *nodal* shape functions. For a scalar problem, the set $\mathcal{N}_i$ contains one shape function which has value one at $\hat{v}_i$ and is zero on the opposite side(s).

The spans of $\mathcal{I}$, Γ_i, $\mathcal{N}_i$ will be denoted by $\tilde{\mathcal{I}}$, $\tilde{\Gamma}_i$, $\tilde{\mathcal{N}}_i$. The above definition does not imply a unique set of shape functions. There are many different ways to create shape functions satisfying the above conditions and spanning the same space. We refer to [3, 17] for details.

Let $M \subset \Re^2$ be a domain. Then by $H^k(M)$, $k > 0$ an integer, we denote the standard Sobolev space on M, and by $\|\cdot\|_{k,M}$ (resp. $|\cdot|_{k,M}$) the norm (resp. seminorm). On

$I = (a,b)$ we also introduce the Sobolev space with fractional index by interpolation by the K-method [6], $H^{1/2}(I) = (H^0(I), H^1(I))_{1/2}$ with the norm $\|\cdot\|_{1/2,I}$ and the space $H_{00}^{1/2} = (H^0(I), H_0^1(I))_{1/2}$ with the norm ${}_0\|\cdot\|_{1/2,I}$. Further, by $H^{1/2}(\partial K)$ we denote the space of all traces of $u \in H^1(K)$ and $\|u\|_{1/2,\partial K} = \inf \|v\|_{1,K}$ where the infimum is taken over all $v \in H^1(K)$ for which $v = u$ on ∂K. We have

$$\|u\|_{1/2,I} \approx \left(\int_a^b \int_a^b \left(\frac{u(x) - u(y)}{x - y} \right)^2 dx dy \right)^{1/2},$$

and

$${}_0\|u\|_{1/2,I}^2 \approx \|u\|_{1/2,I}^2 + 2\int_a^b \frac{u^2(x)}{(x-a)} dx + 2\int_a^b \frac{u^2(x)}{(b-x)} dx \approx \|u\|_{1/2,I}^2 + \int_a^b \frac{u^2(x)\, dx}{(x-a)(x-b)},$$

where $\approx$ denotes the usual equivalence of norms. Let γ_i be the sides of K, $i = 1,2,3$ (resp. $i = 1,2,3,4$) and v_i be the vertices of K such that v_i is common for $\gamma_{j(i)}$ and $\gamma_{l(i)}$. Then it is well known that

$$\|u\|_{1/2,\partial K}^2 \approx \sum_{i=1}^n \|u\|_{1/2,\gamma_i}^2 + \sum_{i=1}^n \int_0^1 \frac{\left(u_{j(i)}(t) - u_{l(i)}(t)\right)^2}{|t|} dt$$

where the $u_j(i)$ denote the restriction of u to $\gamma_j(i)$ and t is the distance to v_i. It is also well known that if $u \in H^{1/2}(\partial \hat{K})$ then there exists $U \in H^1(\hat{K})$ such that

$$\|U\|_{H^1(\hat{K})} \leq C\|u\|_{1/2,\partial\hat{K}}.$$

In this paper we shall be interested in the following model problem: Find u such that

$$-\Delta u = f_1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial^1\Omega, \quad \frac{\partial u}{\partial n} = f_2 \text{ on } \partial^2\Omega \tag{2.2}$$

with $\partial^1\Omega \bigcup \partial^2\Omega = \partial\Omega$. We shall assume that $\partial^1\Omega \neq \emptyset$ and $\partial^1\Omega$ and $\partial^2\Omega$ consist of entire sides of $\partial\Omega$. Note that the assumption that $\partial^1\Omega \neq \emptyset$ is only for the sake of simplicity and is not essential.

We shall understand problem (2.2) in the usual weak form. To this end let

$$H = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial^1\Omega\}$$

and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx dy,$$

be the bilinear form defined on $H \times H$. Further let

$$g(v) = \int_\Omega f_1 v dx dy + \int_{\partial^2\Omega} f_2 v ds$$

be a bounded linear functional on H. Then by the weak solution of (2.2) we mean u_0 such that

$$u_0 \in H: \quad a(u_0, v) = g(v)\ \forall v \in H. \tag{2.3}$$

The solution u_0 exists and is unique.

The finite element solution is defined in the usual way. Let $\mathcal{K}$ be the partition of Ω and

$$V_{\mathcal{K}} = \{u \in H: \quad u|_K \circ F_K \in V_p(\hat{K}),\ K \in \mathcal{K}\}.$$

Then the finite element solution is a function u_{FE} such that

$$u_{FE} \in V_{\mathcal{K}}: \quad a(u_{FE}, v) = g(v), \quad \forall v \in V_{\mathcal{K}}. \tag{2.4}$$

The solution of (2.4) exists and is unique. By integration over the elements we have

$$a(u,v) = \sum_{K \in \mathcal{K}} a_K(u^K, v^K), \qquad a_K(u,v) = \int_K \nabla u^K \nabla v^K \, dx\, dy. \tag{2.5}$$

where $u^K = u|_K$ and $v^K = v|_K$ are the restrictions of u and v to K. Note that the bilinear forms $a_K(u,v)$ are positive semidefinite.

For any $K_i \in \mathcal{K}$ we also have

$$a_K(u,v) = \hat{a}_K(\hat{u}^K, \hat{v}^K)$$

where $\hat{u}^K = u^K \circ F_K \in V_p(\hat{K})$, $\hat{v}^K = v^K \circ F_K \in V_p(\hat{K})$, and $\hat{a}_K$ is defined on the master element $\hat{K}$.

The form $\hat{a}_K$ is different for every $K \in \mathcal{K}$ but

$$\hat{a}_K(\hat{u}^K, \hat{u}^K) \approx \hat{a}(\hat{u}^K, \hat{u}^K),$$

where

$$\hat{a}(\hat{u}^K, \hat{u}^K) = \int_{\hat{K}} |\nabla \hat{u}^K|^2 d\eta d\xi \tag{2.6}$$

and because of (2.1) the equivalence constants are independent of K_i. Hence we do not need, at least for our purposes, to distinguish between the bilinear forms $\hat{a}_K$ and $\hat{a}$. Further, we do not need to make any distinction between the basis (shape) functions on K and $\hat{K}$, or in general between K and $\hat{K}$.

3. Preconditioning by Linear Elements. For any $u \in V_{\mathcal{K}}$ and $K \in \mathcal{K}$ we have the decomposition

$$u^K = \sum_{i=1}^{n} u^K_{\mathbf{v},i} + \sum_{i=1}^{n} u^K_{\mathbf{s},i} + u^K_{\mathbf{I}} \tag{3.1}$$

where $u^K = u|_K$, $u^K_{\mathbf{v},i} \in \tilde{\mathcal{N}}_i$, $u^K_{\mathbf{s},i} \in \tilde{\Gamma}_i$, $u^K_{\mathbf{i}} \in \tilde{\mathcal{I}}$, and $n = 4$ for the rectangle, $n = 3$ for the triangle. (As mentioned in section 2 we do not distinguish between $\hat{K}$ and K). The partition (3.1) is unique. Define

$$c(u,v) = \sum_{K \in \mathcal{K}} c_K(u^K, v^K), \tag{3.2}$$

where

$$c_K(u^K, v^K) = a_K\left(\sum_{j=1}^{n} u^K_{\mathbf{v},j}, \sum_{j=1}^{n} v^K_{\mathbf{v},j}\right) + \sum_{j=1}^{n} a_K(u^K_{\mathbf{s},j}, v^K_{\mathbf{s},j}) + a_K(u^K_{\mathbf{i}}, v^K_{\mathbf{i}}). \tag{3.3}$$

The main goal of this section is to analyze the spectral equivalence of the forms $c(u,v)$ and $a(u,v)$.

LEMMA 3.1. *If*

$$m_1 a_K(u^K, u^K) \le c_K(u^K, u^K) \le m_2 a_K(u^K, u^K) \quad \forall K \in \mathcal{K}\ \forall u^K \in V_p(K),$$

where m_1 and m_2 are independent of $K \in \mathcal{K}$ and u, then

$$m_1 a(u,u) \le c(u,u) \le m_2 a(u,u), \quad \forall u \in V.$$

Proof. Sum over all elements $K \in \mathcal{K}$. □

The following lemma shows that to bound the condition number, it is enough to bound the energy of the terms of the decomposition (3.1).

LEMMA 3.2. *For any $u^K \in V_p(K)$ let*

$$\Big|\sum_{j=1}^{n} u^K_{\mathbf{v},j}\Big|^2_{1,K} \le b_1 |u^K|^2_{1,K}, \tag{3.4}$$

$$|u^K_{\mathbf{s},j}|^2_{1,K} \le b_2 |u^K|^2_{1,K}, \quad j = 1, \dots, n, \tag{3.5}$$

$$|u^K_{\mathbf{i}}|^2_{1,K} \le b_3 |u^K|^2_{1,K}, \quad j = 1, \dots, n. \tag{3.6}$$

Then it holds that

$$m_1 a_K(u^K, u^K) \le c_K(u^K, u^K) \le m_2 a_K(u^K, u^K) \tag{3.7}$$

with $m_2/m_1 \le (n+2)(b_1 + n b_2 + b_3)$.

Proof. We have from the Cauchy-Schwarz inequality

$$\begin{aligned} a_K(u^K, u^K) &= a\Big(\sum_{i=1}^{n} u^K_{\mathbf{v},i} + \sum_{i=1}^{n} u^K_{\mathbf{s},i} + u^K_{\mathbf{i},i}, \sum_{i=1}^{n} u^K_{\mathbf{v},i} + \sum_{i=1}^{n} u^K_{\mathbf{s},i} + u_{\mathbf{i},i}\Big) \\ &\le (n+2)\left(a_K\Big(\sum_{i=1}^{n} u^K_{\mathbf{v},i}, \sum_{i=1}^{n} u^K_{\mathbf{v},i}\Big) + \sum_{i=1}^{n} a_K(u^K_{\mathbf{s},i}, u^K_{\mathbf{s},i}) + a_K(u^K_{\mathbf{i},i}, u^K_{\mathbf{i},i})\right) \\ &= (n+2) c_K(u^K, u^K), \end{aligned}$$

so $m_1 \geq 1/(n+2)$. Further, using (3.4)-(3.6), we obtain $c_K(u^K, u^K) \leq (b_1 + nb_2 + b_3)|u^K|^2_{1,K}$. $\square$

Let us now bound the asymptotic behavior as $p \to \infty$ of the constants b_i for some particular cases of the decomposition (3.1) generated by specific spaces of shape functions.

LEMMA 3.3. *Let $V_p(\hat{K}) = \mathcal{P}^1_p(\hat{T})$ or $\mathcal{P}^2_p(\hat{Q})$, $\hat{\mathcal{N}}_i$ be the set of linear functions when $\hat{K} = \hat{T}$ and bilinear when $\hat{K} = \hat{Q}$, and for all $i = 1, \ldots, n$,*

$$\hat{a}(u, v) = 0 \quad \forall u \in \tilde{\Gamma}, v \in \tilde{\mathcal{I}}, \tag{3.8}$$

where $\hat{a}$ is defined by (2.6). Then (3.4)-(3.6) hold with

$$b_i \leq C(1 + \log^2 p),$$

where C is independent of p.

Proof. Let $u \in V_p(\hat{K})$ and $\bar{u} = u + \lambda$, $\lambda \in \Re$. Then $\sum_{j=1}^n \bar{u}_{\mathbf{v},j} = (\sum_{j=1}^n u_{\mathbf{v},j}) + \lambda$ and $\bar{u}_{\mathbf{s},j} = u_{\mathbf{s},j}$, $\bar{u}_{\mathbf{i}} = u_{\mathbf{i}}$. Because the seminorm $|\cdot|_{1,K}$ and the norm in the factorspace $H^1(\hat{K})/\mathcal{P}_0(\hat{K})$ are equivalent, we may assume that

$$\|u\|_{1,\hat{K}} \leq C|u|_{1,\hat{K}}.$$

Because $\|u\|_{1/2,\hat{\gamma}_i} \leq C\|u\|_{1,\hat{K}}$, applying Theorem 6.2 we obtain

$$\|u_{\mathbf{v},j}\|^2_{1,\hat{K}} \leq C(1 + \log p)|u|^2_{1,\hat{K}}. \tag{3.9}$$

Thus we obtain $b_1 \leq C(1 + \log p)$ and we have for $u_1 = u - \sum_{j=1}^n u_{\mathbf{v},j}$ that

$$\|u_1\|^2_{1,\hat{K}} \leq C(1 + \log p)\|u\|^2_{1,\hat{K}},$$

and using Theorem 6.2 once more we obtain

$$\|u_1\|^2_{L^\infty(\partial\hat{K})} \leq C(1 + \log p)\|u_1\|^2_{1,\hat{K}}.$$

Hence by Theorem 6.6,

$$_0\|u_1\|^2_{1/2,\hat{\gamma}_i} \leq C(1 + \log^2 p)\|u\|^2_{1,\hat{K}}$$

for all sides $\hat{\gamma}_i$. By Theorem 7.4 if $\hat{K} = \hat{T}$ and by Theorem 7.5 if $\hat{K} = \hat{Q}$ there exists $\tilde{u}_{\mathbf{s},j} \in \tilde{\Gamma}_i + \tilde{\mathcal{I}}$ such that $\tilde{u}_{\mathbf{s},j} = u_1$ on $\hat{\gamma}_j$ and

$$\|\tilde{u}_{\mathbf{s},j}\|^2_{1,\hat{K}} \leq C\,_0\|u_1\|^2_{1/2,\hat{\gamma}_j} \leq C(1 + \log^2 p)|u|^2_{1,\hat{K}}.$$

From condition (3.8) we conclude that there also exists $u_{\mathbf{s},j} \in \tilde{\Gamma}_j$ such that

$$\|u_{\mathbf{s},j}\|_{1,\hat{K}} \leq C(1 + \log^2 p)|u_{1,\hat{K}}|$$

and $u_{\mathbf{s},j} = u_1$ on $\hat{\gamma}_j$. Therefore we obtain $b_2 \leq C(1 + \log^2 p)$. Obviously $u_{\mathbf{i}} = u_1 - \sum_{j=1}^n u_{\mathbf{s},j}$ and hence $b_3 \leq C(1 + \log^2 p)$ as well. $\square$

LEMMA 3.4. *Let $V_p(K) = \mathcal{P}_p^3(\hat{Q})$ and let all other assumptions of Lemma 3.3 hold. Then (3.4)–(3.6) hold with $b_i \le Cp^2(1 + \log^2 p)$.*

Proof. The proof is analogous to that above using Theorem 7.6 instead of Theorem 7.5. □

We are now ready for the main result of this section.

THEOREM 3.5. *Let the assumptions of Lemmas 3.3 and 3.4 hold. Then*

$$m_1 a(u,u) \le c(u,u) \le m_2 a(u,u)$$

holds for any $u \in V_{\mathcal{K}}$ with

$$m_2/m_1 \le C(1 + \log^2 p) \tag{3.10}$$

if $V_p(K) = \mathcal{P}_p^1(T)$ or $V_p(K) = \mathcal{P}_p^2(Q)$, $K \in \mathcal{K}$ and

$$m_2/m_1 \le Cp^2(1 + \log^2 p) \tag{3.11}$$

if $V_p(K) \in \mathcal{P}_p^3(Q)$ for some $K \in \mathcal{K}$. The constants C in (3.10) and (3.11) are independent of p and $\mathcal{K}$, and the bound (3.10) cannot be asymptotically improved.

Proof. Combining Lemmas 3.1 - 3.4, we obtain (3.10) and (3.11). To show that the bound (3.10) cannot be asymptotically improved, we take the function function v from the second part of the proof of Theorem 6.2, map it on each side of K and extend to a function u with minimal energy in $V_p(K)$. Then

$$|u|_{1,K}^2 \approx \log p,$$

but it holds for the side components that

$$\sum_{j=1}^{n} |u_{\mathbf{s},j}^K|_{1,K} \approx {}_0\|v\|_{1/2,(0,\pi)}^2 \approx \log^3 p.$$

Consequently, $m_2 \ge (1/C)\log^2 p$; but it is easy to see that $m_1 \le 1$. □

Realizing that the energy of the nodal components was bounded by a $C \log p$, the above proof also shows that the growth of the condition number as $p \to \infty$ is in this case primarily due to the coupling between adjacent sides (and not, for example, between the nodal and side shape functions).

Let us note that we conjecture that (3.11) is too pessimistic, and that in fact the result should be the same as (3.10). This conjecture is supported by numerical experiments.

The solution operator for the problem

$$u \in V_{\mathcal{K}}: \quad c(u,v) = g(v), \quad \forall v \in V_{\mathcal{K}} \tag{3.12}$$

serves now as the preconditioner. It is easy to see that problem (3.12) has a unique solution, since $c(u,u)$ is symmetric and positive definite.

Using this preconditioner we can apply the conjugate gradient method. One such modified preconditioner, which is very natural for existing p-version finite element programs such as PROBE [20], will be studied in the next section. Problem (3.12) is obviously much more easily solved than the original problem and the procedure is highly parallelizable.

4. Implementational Aspect and Numerical Experiments. Theorem 3.5 is the basis for various versions of the preconditioned conjugate gradient method which can be asymptotically equivalent yet different in practical performance. For various aspects we refer to [2, 1, 3, 14, 17, 16, 23]. The method has the following essential steps:

a) Construction of the standard set of shape functions, i.e., the sets $\mathcal{N}_i$, Γ_i, $\mathcal{I}$. Here various practical considerations play an important role, for example the hierarchical character of the functions. For the discussion of the design of $\mathcal{N}_i$, Γ_i, and $\mathcal{I}$, we refer to [3].

b) Transformation of the sets Γ_i of the shape functions to a new set Γ_i^* which satisfies (3.8), while preserving the span of all basis functions. This transformation can be based on the standard form $\hat{a}(\hat{u}^K, \hat{v}^K)$ (and hence made only once on the reference element) or on the actual form $a_K(u^K, v^K)$ made separately (in parallel) on every element. The transformation can be made by elimination (condensation) of the internal shape functions and in the latter case decreases the size of the global stiffness matrix on which we iterate. The transformed shape functions are scaled (normalized) and also orthonormalized in Γ_i^*. Then the stiffness matrix corresponding to the preconditioning form $c(u, v)$ is diagonal except for a diagonal block corresponding to $p = 1$, and the iterations are very simple. It is also possible to choose basis functions on the reference element so that this transformation can be avoided, see [17]. For high p the transformation is relatively expensive when the actual form $a(u, v)$ is used but is fully parallel on the element level. Furthermore, the transformation approach is natural for an existing p-version finite element code such as PROBE.

c) For $p = 1$ preconditioning and conjugate gradient iteration either the global stiffness matrix can be assembled or the iterations can be made using local stiffness matrices. The $p = 1$ preconditioning is relatively inexpensive for higher p.

Various other aspects play an important role in the practical performance of the algorithm. We shall not enter into details here but shall display a numerical example based on one of the versions of the method and implemented on Alliant FX/8[5]. Let us consider problem (2.2) on $\Omega = (-1, 1) \times (-1, 1)$ with the partition $\mathcal{K}$ into $n \times n$ identical squares. We shall further assume that $\partial^2\Omega = \partial\Omega$. The global stiffness matrix is then singular with a simple zero eigenvalue and a constant eigenfunction.

Let us first consider the case where the set $\mathcal{P}_p^3(Q)$ is used on every element of the partition. This set is used in the program PROBE [5, 20].

1) The one element sets $\mathcal{N}_i$ of nodal functions, each consisting of the the usual bilinear function N_i, defined by

$$N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta),$$
$$N_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta),$$

[5] Computational support was provided by the Advanced Computing Research Facility at the Argonne National Laboratory.

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta),$$
$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta).$$

2) The sets Γ_i of the side shape functions. There are $p-1$ shape functions associated with every side $\gamma_i, i = 1,2,3,4$. These are defined as

$$N_i^{[1]}(\xi,\eta) = \frac{1}{2}(1-\eta)\Phi_i(\xi),\ i = 1,2,\ldots,p-1,$$
$$N_i^{[2]}(\xi,\eta) = \frac{1}{2}(1+\xi)\Phi_i(\eta),\ i = 1,2,\ldots,p-1,$$
$$N_i^{[3]}(\xi,\eta) = \frac{(-1)^i}{2}(1+\eta)\Phi_i(\xi),\ i = 1,2,\ldots,p-1,$$
$$N_i^{[4]}(\xi,\eta) = \frac{(-1)^i}{2}(1-\xi)\Phi_i(\eta),\ i = 1,2,\ldots,p-1,$$

where

$$\Phi_i(\xi) = \sqrt{\frac{2i-1}{2}}\int_{-1}^{\xi} P_i(t)dt$$

and $P_j(t)$ is the Legendre polynomial of degree j. The term $(-1)^i$ is needed in $N_i^{[3]}$ and $N_i^{[4]}$ to obtain invariance with respect to rotation of coordinates.

3) The set $\mathcal{I}$ of the internal shape functions. For $p \geq 4$ there are $(p-2)(p-3)/2$ internal shape functions defined as

$$N_{i,j}^{0}(\xi,\eta) = (1-\xi^2)(1-\eta^2)P_i(\xi)P_j(\eta),\ 0 \leq (i+j) \leq p-4. \tag{4.1}$$

For example, if $p = 8$ there are 47 shape functions, consisting of 4 nodal, 28 side and 15 internal shape functions.

This set of shape functions is hierarchical, which is important in practical considerations. In the case of the set $\mathcal{P}_p^2(Q)$ the set $\mathcal{I}$ is expanded so that it contains the functions from (4.1) for all i, j, $0 \leq i,j \leq p-2$. See [2, 3] for details.

In our numerical experiment, we use the following approach:

1) Using the above shape functions we create the local stiffness matrix. We shall simulate the general case where the local stiffness matrices are different and compute them separately in parallel. We further consider two variants:
   a) Preconditioning by elimination of internal shape functions and diagonal scaling of the resulting reduced matrix. Because this reduced matrix is in fact the stiffness matrix with original nodal functions and new side shape functions satisfying (3.8), this corresponds to the preconditioner (3.3) with form a in the second term replaced by the form corresponding to the diagonal of the reduced matrix. To implement (3.3) completely, one can orthogonalize the new side shape functions in the energy inner product, but the condition number would be further reduced only slightly [3, 16].

TABLE 4.1

Number of iterations to reduce the error in the energy norm by factor of 10^{-4} for $n = 2$ (4 elements).

p	Set $\mathcal{P}_p^2(Q)$: Preconditioning by elimination of internal shape functions	Set $\mathcal{P}_p^2(Q)$: No elimination	Set $\mathcal{P}_p^3(Q)$: Preconditioning by elimination of internal shape functions	Set $\mathcal{P}_p^3(Q)$: No elimination
2	12	16	8	8
4	17	31	13	19
6	20	41	16	26
8	22	51	18	32
10	24	59	19	39
12	26	66	20	45
14	27	74	21	50
16	29	79	22	54

b) No elimination is made, the interior shape functions are scaled to obtain ones on the diagonal. This corresponds to the definition of the decomposition from original shape functions, and using the diagonal forms obtained from these shape functions to replace a in (3.3) for sides and interiors.

2) The conjugate gradient method is used with preconditioning (3.3) without assembling the global stiffness matrix. Except for the solution of the problem with $p = 1$ in every iteration, all computations are performed element by element. To measure the convergence, we shall only consider the case with zero exact solution and nonzero random starting vector. We have run our tests for the case $\partial'\Omega = \emptyset$, which leads to singular stiffness matrix with constant eigenfuction. The process was thus adapted by including orthogonal projections onto the complement of the nullspace. Direct solution of the problem for $p = 1$ was done by band LU decomposition and elimination of internal basis functions by full matrix Choleski decomposition. The modified side shape functions have not been orthogonalized.

Table 4.1 shows the number of iterations required to reduce the original error (measured in the energy norm) by a factor of 10^{-4}. We consider the case $n = 2$ (i.e. 4 elements). The results of the previous section indicate that the number of iterations should grow at most as $\log p$ in the case of the set $\mathcal{P}_p^2(\hat{Q})$ and at most as $p \log p$ for the case $\mathcal{P}_p^3(\hat{Q})$, with the conjecture that the growth is only $\log p$. In Fig. 4.1, we show the relation between p and the number of iterations in semilog scale. We see that the case $\mathcal{P}_p^3(\hat{Q})$ needs fewer iterations than $\mathcal{P}_p^2(\hat{Q})$, although the proof is still open. In both cases the growth is $\log p$ for p in practical ranges. (The growth $\log p$ would lead to a straight line in Figure 4.1). Fig. 4.2 shows, in loglog scale, the growth of the number of iterations in the case when no preconditioning by elimination is made. We see that the number of iterations grows about as $p^{3/2}$. This is related to the growth of the condition number of the local stiffness matrix as $O(p^3)$.

To compare the practical potential of both varians, we have to realize that the number of iterations is not solely essential for the effectiveness of the method, because

TABLE 4.2

Timing in seconds on Alliant FX/8 using 4 processors for $n = 2$ (4 elements)

p	Set $\mathcal{P}_p^2(Q)$		Set $\mathcal{P}^3(Q)$	
	Preconditioning by elimination of internal shape functions	No elimination	Preconditioning by elimination of internal shape functions	No elimination
2	0.465 0.018 0.433	0.576 0.011 0.548	0.294 0.010 0.270	0.293 0.010 0.267
4	0.673 0.051 0.610	1.150 0.040 1.098	0.463 0.020 0.436	0.671 0.019 0.639
6	0.969 0.248 0.781	1.791 0.127 1.651	0.743 0.063 0.633	1.131 0.052 1.062
8	1.762 0.946 0.805	3.186 0.327 2.345	0.852 0.206 0.663	1.403 0.116 1.274
10	3.976 2.857 1.092	6.237 0.729 5.483	1.280 0.556 0.709	2.142 0.249 1.879
12	9.263 7.353 1.864	12.498 1.420 11.032	2.173 1.328 0.829	3.344 0.468 2.856
14	20.368 17.088 3.200	24.498 2.535 21.875	3.971 2.866 1.078	5.500 0.820 4.655
16	41.082 35.516 5.414	42.245 4.201 37.893	7.250 5.679 1.534	9.062 1.346 7.676

Legend:	Total time Local stiffness time Conjugate gradients time

TABLE 4.3

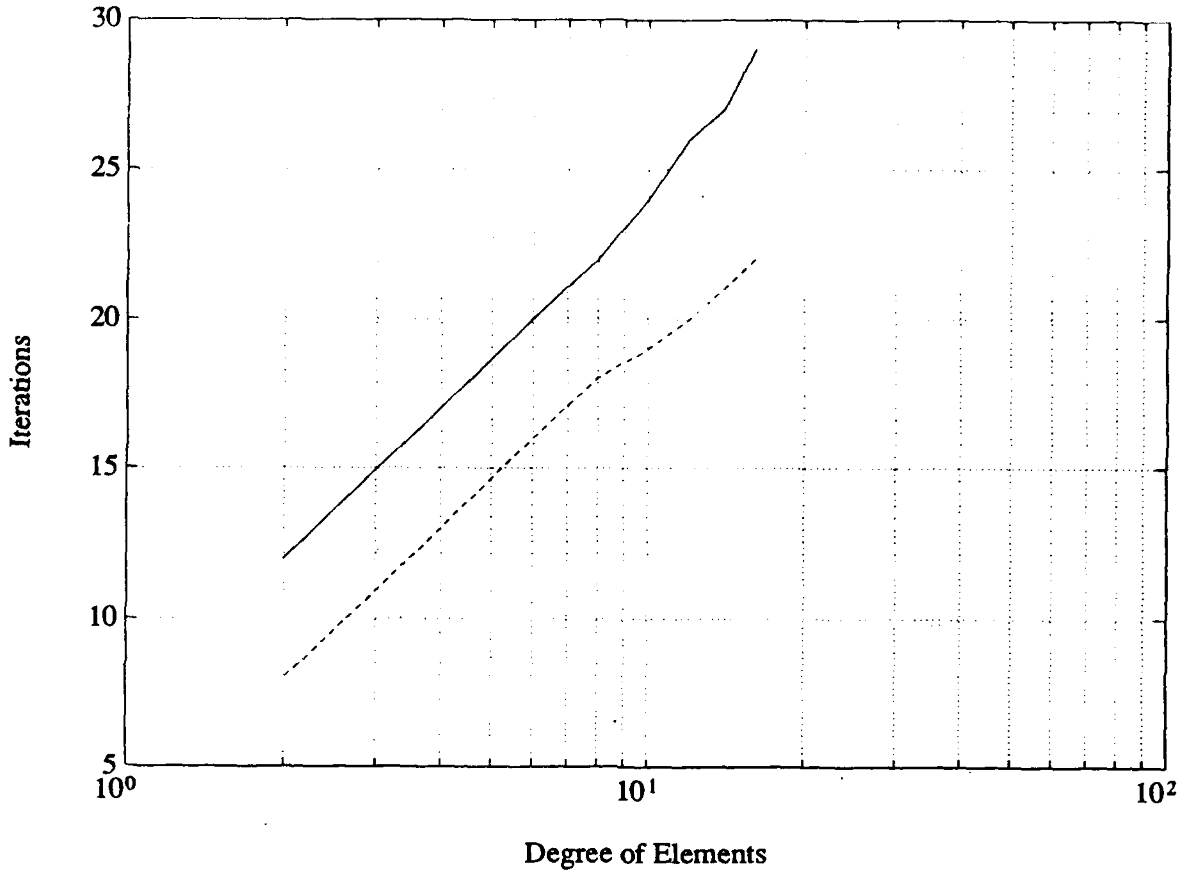
Number of iterations and timing in seconds on Alliant with 8 processors for the set $\mathcal{P}_p^{n,1}(Q)$

p	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 10$
2	10 0.313 0.032 0.263	11 0.473 0.035 0.401	10 0.455 0.125 0.298	10 0.659 0.252 0.368	10 0.560 0.040 0.471	10 0.754 0.146 0.531
4	14 0.458 0.042 0.399	14 0.634 0.071 0.537	14 0.726 0.198 0.498	14 0.972 0.322 0.613	14 0.990 0.158 0.753	14 1.289 0.307 0.896
6	16 0.643 0.133 0.491	16 0.953 0.255 0.666	16 1.112 0.334 0.698	16 1.532 0.504 0.924	16 1.614 0.506 1.049	16 2.625 0.844 1.662
8	17 1.224 0.429 0.769	17 1.740 0.824 0.877	17 2.192 1.106 1.044	17 2.844 1.538 1.260	17 3.211 1.650 1.486	17 5.110 2.732 2.265
10	19 2.191 1.138 1.027	19 3.740 2.268 1.431	19 4.781 2.913 1.826	19 6.592 4.064 2.468	19 7.589 4.573 2.933	19 11.664 7.412 4.140
12	21 4.536 2.747 1.746	21 8.406 5.433 2.895	21 10.788 6.876 3.832	21 15.431 9.653 5.693	20 17.701 10.930 6.632	20 28.187 17.742 10.286
14	22 10.036 5.998 3.913	22 17.871 11.854 5.845	22 22.824 14.872 7.702	22 31.812 20.779 10.710	22 39.008 23.669 14.857	22 61.259 38.391 22.123
16	23 18.902 11.931 6.761	23 34.623 23.746 10.545	23 44.622 29.920 14.232	24 61.470 41.750 19.135	23 74.892 47.458 26.547	23 120.481 77.899 41.193

Legend:

Number of iterations Total time Local stiffness time Conjugate gradient time

FIG. 4.1. *Number of iterations after elimination of interior*



Full line is the number of iterations for the space $\mathcal{P}_p^2$, dashed line for the space $\mathcal{P}_p^3$.

the preconditioning by elimination of interior is expensive. In the table 4.2 we show the timing on Alliant FX/8 with 4 processors (i.e., one per element). We report

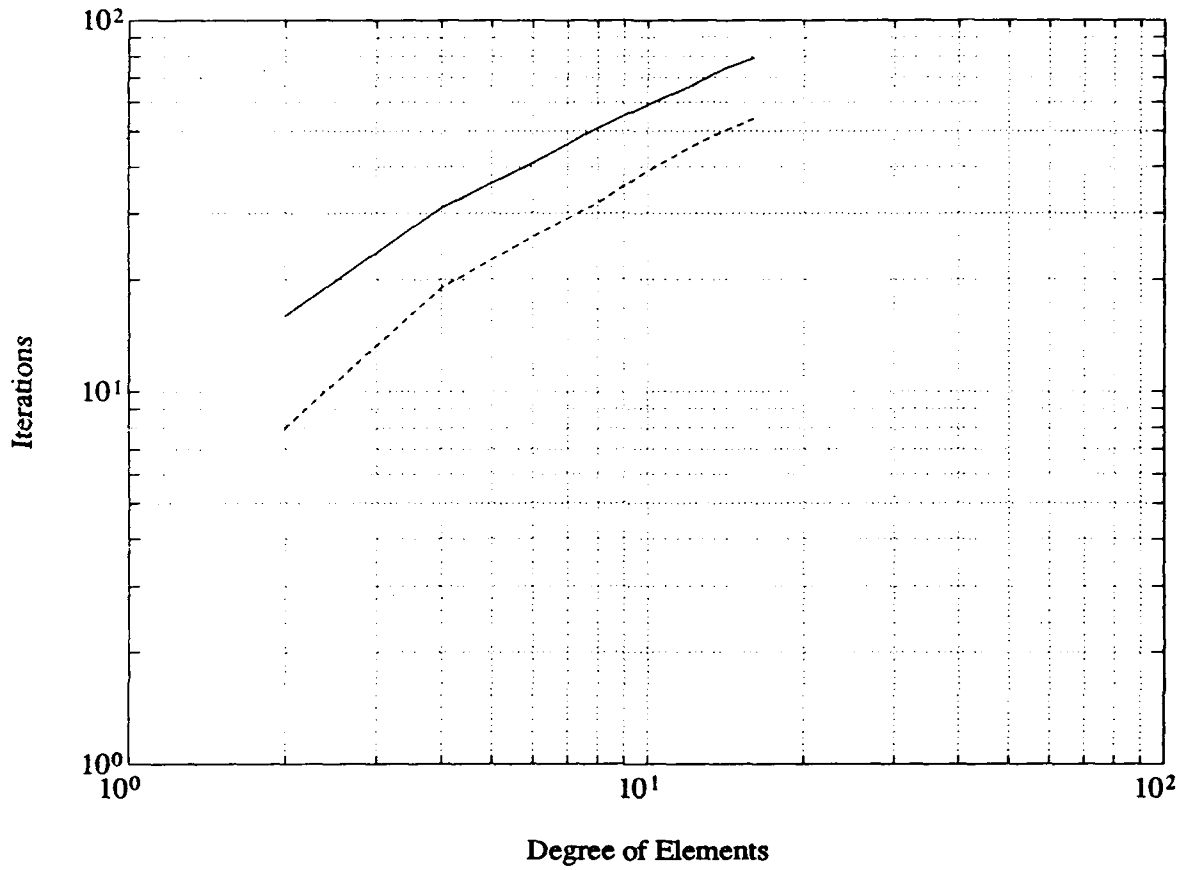
a) the total time,

b) the time for the local stiffness matrix computations, elimination and scaling,

c) the time for the conjugate gradient method.

From table 4.2, we clearly see the timing of the main parts of the computation. In the case where no elimination is made, the local stiffness matrix time consists only of the matrix construction and scaling, while in the case of elimination it also includes the time for elimination which, for high p, is the main part of the total time. Comparing these times we see that the construction of the local stiffness matrix is not overly expensive. Further we see that the use of the set $\mathcal{P}_p^3(\hat{Q})$ is clearly superior to that of $\mathcal{P}_p^2(\hat{Q})$. The set $\mathcal{P}_p^2(\hat{Q})$ will be more accurate than $\mathcal{P}_p^3(\hat{Q})$ for the same p, but $\mathcal{P}_p^2(\hat{Q})$ has more basis functions and a

FIG. 4.2. *Number of iterations without elimination of interior*



Full line is the number of iterations for the space $\mathcal{P}_p^2$, dashed line for the space $\mathcal{P}_p^3$.

greater increase in accuracy will be obtained by increasing p in $\mathcal{P}_p^3(\hat{Q})$. (For example $p = 11$ for $\mathcal{P}_p^2(\hat{Q})$ is comparable with $p = 16$ for $\mathcal{P}_p^3(\hat{Q})$.)

So far we have presented the data for the case $n = 2$ (i.e. 4 elements). Table 4.3 reports the number of iterations as functions of p and n for the set $\mathcal{P}_p^3(\hat{Q})$ and the timing as in table 4.2. We see that the number of iterations is independent of n and the growth with p is the same as in table 4.1. We report here condensed data only; for a detailed breakdown of the timing and an exact description of the tests, we refer to [2, 23]. Here we mention only that the total time does not equal to the sum of local stiffness time and CG time. This difference includes the time for the LU decomposition for $p = 1$ as well as various communications and bookkeeping operations. Note in Table 4.3 that the local stiffness time for large p is almost proportional to the number of elements (see, e.g. $p = 16$, $n = 4, 8$), while for low p other factors prevail. Because in our model problem the local stiffness matrices are identical, we could compute the local stiffness matrices only once by each processor. The local stiffness time would then decrease by the factor $n^2/8$. The LU decomposition for $p = 1$ is of order 3% of the total time for high p and 10% of total time for low p. Finally, we mention that for $p = 16$ and $n = 10$, the size of the global stiffness matrix (number of degrees of freedom) was 12521.

We conclude this section with several remarks based on detailed computational results as reported in [23]. The method is well parallelizable and the observed speedup is very high. However, the speedup is different for various parts of the procedure. The local stiffness matrix computation is completely independent and thus the speedup is more than 95%. The conjugate gradient iterations, although parallelizable element by element, require considerable communication between elements and thus the speedup is smaller. The entire computation has speedup of order at least 85%, depending on the number of elements and the degree p.

We expect that that the speed up will be essentially the same for parallel computers with distributed memory because of good load balancing. The stated timings also allow us to roughly estimate the times for other variations of the approach, assuming that the same number of iterations are required. The stopping criterion 10^{-4} is realistic because the discretization error, measured in the energy norm, is, in practice, larger than 1% and hence an additional error of 0.01% is fully acceptable. The reported times are for elements which are not distorted. We can expect that strong distortion of the elements will strongly influence the number of iterations required; see Mandel [17, 16] for related condition numbers in three dimensions and curved elements.

There are many other obvious versions and variations of the implementation. We have shown one possibility, which can be easily implemented as a part of an existing p-version code.

5. Preconditioning by an Auxiliary System. The method of Section 3 cannot be applied successfully to three-dimensional problems because the values of $\|u^K_{\mathbf{v},j}\|_{1,K}$ cannot be bounded by a power of $\log p$ times $\|u\|_{1,K}$ independently of u as $p \to \infty$. We shall introduce another preconditioning system which has been successfully applied in three dimensions, see Mandel [17]. We shall analyze the analogous two dimensional procedure. In order to analyze the three-dimensional procedure, we would need the appropriate extension theorems in three dimensions; the proof is then completely analogous. The method we obtain is related to that of Bramble, Pasciak, and Schatz [7] for the h-version.

For $u \in V_{\mathcal{K}}$ and $K \in \mathcal{K}$ we write as before in (3.1),

$$u^K = \sum_{i=1}^{n} u^K_{\mathbf{v},i} + \sum_{i=1}^{n} u^K_{\mathbf{s},i} + u^K_{\mathbf{i}}, \tag{5.1}$$

where $u^K_{\mathbf{v},i} \in \tilde{\mathcal{N}}_i$, $u^K_{\mathbf{s},i} \in \tilde{\Gamma}_i$, and $u^K_{\mathbf{i}} \in \tilde{\mathcal{I}}$. The definitions of the spaces $\tilde{\mathcal{N}}_i$, $\tilde{\Gamma}_i$, and $\tilde{\mathcal{I}}$ will be different here. In Section 3, it was essential that the span of $\mathcal{N}_i$, $i = 1, \ldots, n$, contains the constant functions. This is no longer required here; instead, we take care of the constant component separately. To this end, let Λ be the space of functions which are constant on each $K \in \mathcal{K}$. Then on $(V_{\mathcal{K}} \times \Lambda) \times (V_{\mathcal{K}} \times \Lambda)$, we define the form

$$d(u, \lambda; v, \mu) = \sum_{K \in \mathcal{K}} d_K(u^K, \lambda^K; v^K, \mu^K),$$

where

$$d_K(u^K, \lambda^K; v^K, \mu^K) = d_K^*(u^K - \lambda^K, v^K - \mu^K),$$

and

$$d_K^*(u, v) = \sum_{j=1}^{n} a_K(u_{\mathbf{v},j}^K, v_{\mathbf{v},j}^K) + \sum_{j=1}^{n} a_K(u_{\mathbf{s},j}^K, v_{\mathbf{s},j}^K) + a_K(u_I^K, v_I^K).$$

The solution operator for the problem

$$u \in V: \quad d(u, \lambda; v, \mu) = g(v), \quad \forall v \in V_{\mathcal{K}},\ \mu \in \Lambda \tag{5.2}$$

will now, analogously to (3.8), define the preconditioner (using u only). Existence of the solution operator follows easily from the observation that $d(u, \lambda; u, \lambda) \geq 0$ with equality only if $u = 0$ and $\lambda = 0$. The system (5.2) obviously splits into

$$d_K^*(u^K - \lambda^K, 1) = 0 \tag{5.3}$$

and

$$\sum_{K \in \mathcal{K}} d_K^*(u^K - \lambda^K, v^K) = g(v), \quad \forall v \in V_{\mathcal{K}} \tag{5.4}$$

See [17] for more details about the solution of (5.3) and (5.4). Now we have, analogously to Lemma 3.1,

LEMMA 5.1. *If*

$$m_1 a_K(u^K, u^K) \leq d_K(u^K, \lambda^K(u^K); u^K, \lambda^K(u^K)) \leq m_2 a_K(u^K, u^K)$$

where $\lambda^K(u^K)$ is defined by (5.3) and m_1 and m_2 are independent of $K \in \mathcal{K}$ and u, then

$$m_1 a(u, u) \leq d(u, \lambda(u); u, \lambda(u)) \leq m_2 a(u, u).$$

Further we have

LEMMA 5.2. *Assume that for any $u^K \in V_K$ such that $\int_K u^K dxdy = 0$, it holds that*

$$|u_{\mathbf{v},j}^K|_{1,K}^2 \leq b_1 |u^K|_{1,K}^2, \quad |u_{\mathbf{s},j}^K|_{1,K}^2 \leq b_2 |u^K|_{1,K}^2, \quad |u_I^K|_{1,K}^2 \leq b_3 |u^K|_{1,K}^2. \tag{5.5}$$

Then

$$m_1 a_K(u^K, u^K) \leq d_K(u^K, \lambda(u^K); u^K, \lambda(u^K)) \leq m_2 a_K(u^K, u^K) \tag{5.6}$$

holds for all $u_K \in V_K$ *with* $m_2/m_1 \leq (2n+1)(nb_1 + nb_2 + b_3)$.

Proof. The proof is the same as for Lemma 3.2 observing that for any λ,

$$a_K(u^K - \lambda, u^K - \lambda) = a_K(u^K, u^K)$$

and using the fact that all the terms in (5.6) are invariant with respect to the addition of a constant to u. □

Let us now determine the constants b_i for some particular choices of the shape functions.

LEMMA 5.3. *Let*

a) $V_p(\hat{K}) = \mathcal{P}^1_p(\hat{T})$ *or* $\mathcal{P}^2_p(\hat{Q})$

b) For any $u \in \tilde{\mathcal{N}}_i$, $i = 1, \ldots, n$, *and all sides* $\hat{\gamma}_j$ *adjacent to the vertex* $\hat{v}_j$,

$$\hat{a}(u,v) = 0 \qquad, \forall v \in \tilde{\mathcal{I}} \cup \tilde{\Gamma}_j, \tag{5.7}$$

c) For $i = 1, \ldots, n$,

$$\hat{a}(u,v) = 0 \qquad \forall u \in \tilde{\Gamma}_i, \quad \forall v \in \tilde{\mathcal{I}}. \tag{5.8}$$

Then (5.5) holds with

$$b_i \leq C(1 + \log^2 p), \; i = 1,2,3,$$

where C *is independent of* p.

Proof. First consider the case when $\hat{K} = \hat{T}$ is the reference triangle with the vertices $\hat{v}_i, i = 1,2,3$ and sides $\hat{\gamma}_i, i = 1,2,3$ opposite to $\hat{v}_i$ (see Fig 2.2). Let $u \in V_p(\hat{K})$, $\int_{\hat{K}} u \, d\xi \, d\eta = 0$; then

$$\|u\|_{1,\hat{K}} \leq C|u|_{1,\hat{K}}$$

By Theorem 7.7 and the trace theorem, there exists a function $f_i \in V_p(\hat{K})$ such that $f_i = u$ on $\hat{\gamma}_i$ and $f_i = 0$ at $\hat{v}_i$. Then $\bar{v}_{\mathbf{V},i} = u - f_i$ is zero at $\hat{\gamma}_i$ and $u(v_i) = \bar{v}_{\mathbf{V},i}(v_i)$. Then from (5.7) we have $|u^K_{\mathbf{V},i}|_{1,\hat{K}} \leq |\bar{v}_{\mathbf{V},i}|_{1,\hat{K}}$ and hence

$$|u_{\mathbf{V},i}|_{1,\hat{K}} \leq C|u|_{1,\hat{K}}, \tag{5.9}$$

giving $b_1 \leq C$.

Now for $u_1 = u - \sum_{j=1}^n u_{\mathbf{V},j}$, we have $\|u_1\|_{1,\hat{K}} \leq C\|u\|_{1,\hat{K}}$, and using Theorem 6.5, we get

$$_0\|u_1\|^2_{1/2,\hat{\gamma}_i} \leq C(1 + \log^2 p)\|u\|^2_{1,\hat{K}}.$$

The rest of the proof for this case is analogous to that of Lemma 3.3.

Let us now consider the case $\hat{K} = \hat{Q}$ and the vertex $\hat{v}_1$ (see Figure 2.1). By Corollary 6.3 and Theorem 7.9, there is a function $v \in \mathcal{P}^2_p$ such that $v|_{\hat{\gamma}_1} = v|_{\hat{\gamma}_2} = 0$, $v(\hat{v}_1) = u(\hat{v}_1)$, and

$$\|v\|_{1,\hat{K}} \leq C\|u\|_{1,\hat{K}}.$$

Now we may conclude using (5.7) that

$$\|u_{V,1}\|_{1,\hat{K}} \leq C\|u\|_{1,\hat{K}}$$

and hence $b_1 \leq C$ in (5.5). The above estimate obviously holds for $u_{V,i}$, $i = 2,3,4$ as well.

The proof of the estimate of b_2 and b_3 goes exactly as in in the first case using extensions by Theorem 7.5. □

We have the following bound for the tensor product space.

LEMMA 5.4. *Let $V_p(\hat{Q}) = \mathcal{P}_p^3(\hat{Q})$ and (5.7) and (5.8) hold. Then $|b_i| \leq Cp^2(1+\log^2 p)$.*

Proof. The proof is same as that of Lemma 5.6 for $\hat{K} = \hat{Q}$ using the extension by Theorem 7.6 rather than by Theorem 7.5. □

We are now ready for the main result of this section.

THEOREM 5.5. *Let the assumptions of Lemmas 5.2 and 5.4 hold. Then*

$$m_1 a(u,u) \leq d(u,\lambda(u);u,\lambda(u)) \leq m_2 a(u,u)$$

with

$$m_2/m_1 \leq C(1+\log^2 p) \tag{5.10}$$

if $V_p(\gamma_K) = \mathcal{P}_p^1(\gamma_T)$ or $V_p(\gamma_K) = \mathcal{P}_p^2(\gamma_Q)$ for all $K \in \mathcal{K}$ and

$$m_2/m_1 \leq Cp^2(1+\log^2 p) \tag{5.11}$$

if $V_p(\hat{K}) = \mathcal{P}_p^3(\hat{Q})$ for some $K \in \mathcal{K}$. The constant C in (5.10) and (5.11) is independent of p and $\mathcal{K}$ and the bound (5.10) cannot be asymptotically improved.

Proof. Combining Lemmas 5.1-5.4 we obtain (5.10) and (5.11). The proof that (5.10) is sharp is same as in the proof of Theorem 3.5. □

We conjecture as in Section 3 that the estimate (5.11) is pessimistic and that it can be in fact replaced by (5.10).

6. Polynomial Subspaces of $H^{1/2}$. In this section, we give several results for spaces of polynomials on a segment, which are of interest in themselves. First we prove a discrete Sobolev inequality for the $H^{1/2}$ norm, bounding the pointwise value of a polynomial in terms of its $H^{1/2}$ norms. Then we bound the $H_{00}^{1/2}$ norm of a polynomial which is zero on the endpoints in terms of its $H^{1/2}$ norm. In what follows we have $K = \hat{Q}$ or $K = \hat{T}$, but we use the coordinates x, y instead of ξ, η.

We begin with a lemma which will allow us to consider trigonometric polynomials instead of algebraic ones. Let $I = (-1,+1)$ and $I^* = (0,\pi)$.

LEMMA 6.1. *Define the mapping Ψ by*

$$u \in H^{1/2}(I) \mapsto \Psi u = v, \quad v(\varphi) = u(\cos\varphi), \quad \varphi \in I^*. \tag{6.1}$$

Then Ψ is a linear homeomorphism between $H^{1/2}(I)$ and $H^{1/2}(I^*)$.

If $u \in \mathcal{P}_p(I)$ and $v = \Psi(u)$, then

$$v(\varphi) = \sum_{k=0}^{p} b_k \cos k\varphi \tag{6.2}$$

and

$$\|v\|^2_{1/2,I^*} \approx \sum_{k=0}^{p} b_k^2(k+1). \tag{6.3}$$

Proof. Define

$$\begin{aligned} &\Omega = I^* \times I, \\ &\Phi : (\varphi, \zeta) \to (x, y), \quad x + iy = \cos(\varphi + i\zeta), \\ &Z = \Phi(\Omega). \end{aligned}$$

Because the cosine is a conformal mapping, the Cauchy-Riemann conditions hold,

$$x_\varphi = y_\zeta, \quad x_\zeta = -y_\varphi,$$

and we have the Jacobian of Φ,

$$J = x_\varphi^2 + y_\zeta^2 \neq 0, \quad \forall (\varphi, \zeta) \in \Omega,$$

because $-\sin(\varphi + i\zeta) = \cos'(\varphi + i\zeta) = x_\varphi + iy_\varphi \neq 0$ for all $(\varphi, \zeta) \in \Omega$. Now let u be a smooth function defined on Z. Define

$$\Psi(u) = v(\varphi, \zeta) = u(\Phi(\varphi, \zeta)).$$

Then v is defined on Ω. From the chain rule and Cauchy-Riemann conditions,

$$v_\varphi = u_x x_\varphi + u_y y_\varphi, \qquad v_\zeta = u_x x_\zeta + u_y y_\zeta = -u_x y_\varphi + u_y x_\varphi,$$

and by a simple computation,

$$v_\varphi^2 + v_\zeta^2 = (u_x^2 + u_y^2)(x_\varphi^2 + y_\varphi^2) = (u_x^2 + u_y^2)J.$$

By substitution,

$$|u|^2_{1,Z} = \int_Z (u_x^2 + u_y^2)\, dx\, dy = \int_\Omega (u_x^2 + u_y^2) J \, d\varphi\, d\zeta = \int_\Omega (v_\varphi^2 + v_\zeta^2)\, d\varphi\, d\zeta = |v|^2_{1,\Omega}.$$

Finally, from

$$|u|^2_{0,Z} = \int_Z |u|^2 \, dx\, dy = \int_\Omega |v|^2 J \, d\varphi\, d\zeta$$

and the equivalence of norms

$$\|z\|_{1,\Omega}^2 \sim \left(|z|_{1,\Omega}^2 + \int_\Omega w|z|^2 \, d\varphi \, d\zeta \right)^{1/2}, \qquad z \in H^1(\Omega),$$

which holds for any function $w \in L^\infty(\Omega)$ such that $w \geq 0$ and $\int_\Omega w \, d\varphi \, d\zeta > 0$, it follows that

$$\frac{1}{C}\|\Psi(u)\|_{1,\Omega} \leq \|u\|_{1,Z} \leq C\|\Psi(u)\|_{1,\Omega}, \tag{6.4}$$

with C independent on u. By continuity, (6.4) holds for all $u \in H^1(Z)$.

Now if $u \in H^{1/2}(I)$, we may extend u to $u \in H^1(Z)$ so that $\|u\|_{1,Z} \leq C\|u\|_{1/2,I}$, and it follows from (6.4) and the trace theorem that

$$\|\Psi(u)\|_{1/2,I^*} \leq C\|\Psi(u)\|_{1,\Omega} \leq C\|u\|_{1,Z} \leq C\|u\|_{1/2,I}.$$

The inverse inequality follows similarly using Ψ^{-1} in place of Ψ.

To prove (6.2), it is sufficient to note that we have $\cos^n \varphi = \sum_{k=0}^n a_k \cos n\varphi$ for suitable a_k. Equation (6.3) follows by direct evaluation of Sobolev norms of a Fourier series and by interpolation. □

The following theorem is a discrete Sobolev inequality for polynomial spaces.

THEOREM 6.2. *Let $I = (-1,+1)$, $u \in \mathcal{P}_p(I)$, and $x \in I$. Then*

$$|u(x)| \leq C(1 + \log^{1/2} p)\|u\|_{1/2,I},$$

with C independent on p, u, and x.

This estimate cannot be asymptotically improved, i.e., there is a constant C such that for each $p > 2$ there exists $u_p \in \mathcal{P}_p(I)$ such that $\|v_p\|_{1/2,I} \leq C$ and $|u_p(-1)| \geq \log^{1/2} p$.

Proof. By Lemma 6.1, we can consider instead the case $v \in H^{1/2}(I^*)$,

$$v(\varphi) = \sum_{k=0}^p a_k \cos(\varphi),$$

and $\psi \in I^*$, $x = \cos\psi$. Then by the Cauchy-Schwarz inequality,

$$\begin{aligned} |v(\psi)| &\leq \sum_{k=0}^p |a_k| = \sum_{k=0}^p |a_k|(k+1)^{1/2}\frac{1}{(k+1)^{1/2}} \\ &\leq \left(\sum_{k=0}^p |a_k|^2(k+1)\right)^{1/2}\left(\sum_{k=0}^p \frac{1}{k+1}\right)^{1/2} \leq C\|v\|_{1/2,I^*}(1+\log^{1/2} p). \end{aligned}$$

To show optimality of the estimate, it is enough to choose $a_k = (k+1)^{-1}$ and $x = -1$. □

COROLLARY 6.3. *We have*

$$\|u\|_{L^\infty(\partial K)} \leq C(1 + \log^{1/2} p)\|u\|_{1,K}, \quad \forall u \in \mathcal{P}_p'(K), \tag{6.5}$$

$i = 1, 2$ *or* 3, *where* C *does not depend on* u.

The following lemma extends the results of Lemma 6.1 showing that the mapping Ψ preserves the subspace $H_{00}^{1/2}$.

LEMMA 6.4. *Let* Ψ *be defined as in (6.1). Then* $u \in H_{00}^{1/2}(I)$ *if and only if* $\Psi(u) \in H_{00}^{1/2}(I^*)$, *and* Ψ *is a linear homeomorphism between* $H_{00}^{1/2}(I)$ *and* $H_{00}^{1/2}(I^*)$.

Proof. Let $v = \Psi(u)$. It suffices to use Lemma 6.1 and to note that we have for the additional term in the $H_{00}^{1/2}$ norm that

$$\int_{-1}^{1} \frac{|u^2(x)|}{|x^2-1|}\, dx = \int_{\pi}^{0} \frac{|u(\cos^2\varphi)|^2}{|\cos\varphi - 1|}(-\sin\varphi)\, d\varphi \left\{ \begin{array}{c} \leq C \\ \geq 1/C \end{array} \right\} \int_{0}^{\pi} \frac{|v^2(\varphi)|}{\varphi(\pi-\varphi)} d\varphi,$$

since

$$\frac{\sin\varphi}{1-\cos^2\varphi} \left\{ \begin{array}{c} \leq C \\ \geq 1/C \end{array} \right\} \frac{1}{\varphi(\pi-\varphi)}, \quad \varphi \in (0,\pi).$$

□

THEOREM 6.5. *Let* Z_p *be the space*

$$Z_p = \{u \in \mathcal{P}_p(I) \mid u(-1) = u(+1) = 0\}.$$

Then for all $p > 0$,

$$(6.6) \qquad {}_0\|u\|_{1/2,I} \leq C(1+\log p)\|u\|_{1/2,I}, \quad \forall u \in Z_p,$$

with a constant C *independent on* p *and* u. *The bound (6.6) cannot be asymptotically improved, that is, for every* p, *there exists a function* $v \in Z_p$ *such that* ${}_0\|v\|_{1/2,I}/\|v\|_{1/2,I} \geq C\log p$, $C > 0$ *independently of* p.

Proof. From Lemmas 6.1 and 6.4, it suffices to consider instead the case

$$v(\varphi) = \sum_{k=0}^{p} a_k \cos k\varphi, \quad v(0) = 0.$$

We use the fact that $v(0) = \sum_{k=0}^{p} a_k = 0$ and estimate by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_0^{\pi} \frac{|v(\varphi)|^2}{\varphi}\, d\varphi &= \int_0^{\pi} \frac{1}{\varphi} \left(\sum_{k=1}^{p} a_k(\cos k\varphi - 1) \right)^2 d\varphi \\ &\leq \int_0^{\pi} \frac{1}{\varphi} \left(\sum_{k=1}^{p} |a_k|^2 (k+1) \right) \left(\sum_{k=1}^{p} \frac{(\cos k\varphi - 1)^2}{k+1} \right) d\varphi \\ &\leq C\|v\|^2_{1/2,I^*} \sum_{k=1}^{p} \int_0^{\pi} \frac{(\cos k\varphi - 1)^2}{\varphi(k+1)}\, d\varphi. \end{aligned}$$

But

$$\int_0^{\pi} \frac{(\cos k\varphi - 1)^2}{\varphi} d\varphi = \int_0^{k\pi} \frac{(\cos\psi - 1)^2}{\psi} d\psi \leq C(1+\log k),$$

so

$$\int_0^\pi \frac{|v(\varphi)|^2}{\varphi} d\varphi \le C\|v\|^2_{1/2,I} \cdot \sum_{k=1}^{p} \frac{\log k}{k+1} \le C\|v\|^2_{1/2,I} . C(1+\log^2 p).$$

The bound of the integral $\int_0^\pi \frac{|v(\varphi)|^2}{\pi-\varphi} d\varphi$ when $v(\pi) = 0$ follows by the substitution of φ for $\pi - \varphi$.

To show that the bound (6.6) is sharp, let

$$v(\varphi) = \sum_{k=0}^{p} a_k \cos k\varphi$$

with

$$a_k = \begin{cases} \left.\begin{array}{ll} 1/k, & 2 \le k \le M \\ -\beta/k, & M < k \le p \end{array}\right\} k \text{ even} \\ 0, \quad k \text{ odd or } k = 0. \end{cases}$$

where $M = [\sqrt{p}]$ (integer part) and $\beta = O(1)$ is chosen so that $\sum_{k=0}^{p} a_k = 0$, i.e., $v(0) = v(\pi) = 0$. Now expand v so that

$$v(\varphi) = \sum_{m=1}^{\infty} b_m \sin m\varphi.$$

Since $\|v\|^2_{1/2,(0,\pi)} \approx \log p$, it will suffice to show that

$$_0\|v\|^2_{1/2,(0,\pi)} \approx \sum_{m=1}^{\infty} m b_m^2 \ge \frac{1}{C} \log^3 p.$$

We have

$$b_m = \begin{cases} 0, & m \text{ even}, \\ \frac{2}{\pi} \sum\limits_{\substack{2 \le k \le p \\ k \text{ even}}} \frac{2m}{m^2-k^2} a_k, & m \text{ odd} \end{cases}$$

Then $b_m = (2/\pi)(b_m^{(1)} - \beta b_m^{(2)})$, where

$$b_m^{(1)} = \sum_{\substack{2 \le k \le M \\ k \text{ even}}} \frac{2m}{m^2-k^2} \frac{1}{k}, \qquad b_m^{(2)} = \sum_{\substack{M+1 \le k \le p \\ k \text{ even}}} \frac{2m}{m^2-k^2} \frac{1}{k}.$$

The function $x \mapsto \frac{2m}{(m^2-x^2)x}$ is decreasing on the interval $(2, m/\sqrt{3})$, increasing on the intervals $(m/\sqrt{3}, m)$ and $(m, +\infty)$, and positive on $(2, m)$. Hence,

$$b_m^{(1)} \ge C\Bigg(\sum_{\substack{2 \le k \le [m/\sqrt{3}]-1 \\ k \text{ even}}} \frac{2m}{(m^2-k^2)k} + \sum_{\substack{m+1 \le k \le M \\ k \text{ even}}} \frac{2m}{(m^2-k^2)k} \Bigg)$$

$$\ge C\Bigg(\int_2^{[m/\sqrt{3}]-1} \frac{2m\,dx}{(m^2-x^2)x} + \int_{m+1}^{M} \frac{2m\,dx}{(m^2-x^2)x} + \frac{2m}{(m^2-(m+1)^2)(m+1)} \Bigg).$$

Using the fact that

$$\int \frac{dx}{(m^2 - x^2)x} = \frac{1}{2m^2} \log \left| \frac{x^2}{m^2 - x^2} \right| + C,$$

we have by a simple computation that

$$b_m^{(1)} \geq C \frac{\log m}{m}, \qquad C > 0, \quad 4 \leq m \leq M.$$

Because $b_m^{(2)} < 0$ for $m \leq M$, we may conclude that

$$b_m \geq C \frac{\log m}{m}, \qquad C > 0, \quad 4 \leq m \leq M$$

and thus

$$\sum_{m=1}^{\infty} m b_m^2 \geq C \sum_{m=4}^{[\sqrt{p}]} \frac{\log m}{m^2} \geq C \log^3 p, \qquad C > 0,$$

for all sufficiently large p. □

The next theorem gives a different bound.

THEOREM 6.6. *Let Z_p be as in Theorem 6.5. Then for all $p > 0$,*

$$_0\|u\|_{1/2,I}^2 \leq \|u\|_{1/2,I}^2 + C(1 + \log p)\|u\|_{L^\infty(I)}^2.$$

Proof. It is sufficient to bound the additional term in the $H_{00}^{1/2}$ norm. We have

$$\int_{-1}^{1} \frac{v^2(t)}{1-t}\, dt = \int_{1-1/p^2}^{1} \frac{v^2(t)}{1-t}\, dt + \int_{-1}^{1-1/p^2} \frac{v^2(t)}{1-t}\, dt.$$

By Markov's inequality, cf., [19],

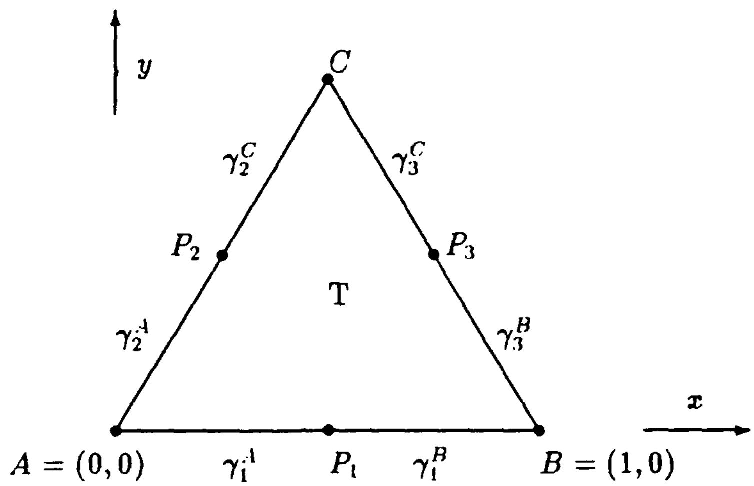
$$\|v'\|_{L^\infty(I)} \leq Cp^2 \|v\|_{L^\infty(I)},$$

so $|v(t)| \leq C(1-t)p^2 \|v\|_{L^\infty(I)}$ and we obtain

$$\int_{-1}^{1} \frac{v^2(t)}{1-t}\, dt \leq C\|v\|_{L^\infty(I)}^2 \left(\int_{1-1/p^2}^{1} (1-t)p^4 dt + \int_{-1}^{1-1/p^2} \frac{1}{1-t} dt \right)$$
$$\leq C \left(\frac{1}{2} + 2\log 2p \right) \|v\|_{L^\infty(I)}^2.$$

The analogous bound on $\int_{-1}^{1} \frac{v^2(t)}{-1-t}\, dt$ follows by substitution. □

FIG. 7.1. *Notation scheme for the triangle*



7. Polynomial Extensions from the Boundary. Let $K = \hat{T}$ or $K = \hat{Q}$. If $f \in H^{1/2}(\partial K)$, then there exists an extension $F \in H^1(K)$ so that $f = F$ on ∂K and

$$\|F\|_{1,K} \leq C\|f\|_{1/2,\partial K}.$$

The main result of this section is that if f is a polynomial of degree p on all sides of K, then F can be chosen to be in $\mathcal{P}_p^i$ $i = 1,2$ if K is a triangle or a square, respectively. This extends previous results from Babuška and Suri [4].

Let us consider the triangle $T = ABC$ as shown in Fig. 7.1. We denote

$$\begin{aligned}
\gamma_1 &= \gamma_1^A \cup \gamma_1^B = \overline{AP_1} \cup \overline{P_1B} = \overline{AB}, \\
\gamma_2 &= \gamma_2^A \cup \gamma_2^C = \overline{AP_2} \cup \overline{P_2C} = \overline{AC}, \\
\gamma_3 &= \gamma_3^B \cup \gamma_3^C = \overline{BP_3} \cup \overline{P_3C} = \overline{BC}.
\end{aligned}$$

Let $f \in \mathcal{P}_p(\gamma_1)$. Then we define

$$F_1^{[f]}(x,y) = \frac{\sqrt{3}}{2y} \int_{x-\frac{y}{\sqrt{3}}}^{x+\frac{y}{\sqrt{3}}} f(t)\,dt. \tag{7.1}$$

The value of F_1 at a point $(x,y) \in T$ depends only on the values f along the segment $\overline{Q_1Q_2}$, $Q_1 = (x - \frac{y}{\sqrt{3}}, 0)$, $Q_2 = (x + \frac{y}{\sqrt{3}}, 0)$. We prove now the following lemma.

LEMMA 7.1. *Let $f \in \mathcal{P}_p(\gamma_1)$ and $F_1^{[f]}(x,y)$ be defined by (7.1). Then*

$$F_1^{[f]}(x,y) \in \mathcal{P}_p^1(T), \tag{7.2}$$

(7.3) $$F_1^{[f]}(x,0) = f(x)$$

(7.4) $$\|F_1^{[f]}\|_{1,T} \le C\|f\|_{1/2,\gamma_1}$$

(7.5) $$\|F_1^{[f]}\|_{k,\gamma_2^A} \le C\|f\|_{k,\gamma_1^A}, \quad 0 \le k \le 1,$$

(7.6) $$\|F_1^{[f]}\|_{k,\gamma_3^B} \le C\|f\|_{k,\gamma_1^B} \quad 0 \le k \le 1,$$

(7.7) $$\|F_1^{[k]}\|_{k,\gamma_2^C} \le C\|f\|_{0,\gamma_1}, \quad 0 \le k \le 1,$$

(7.8) $$\|F_1^{[f]}\|_{k,\gamma_3^C} \le C\|f\|_{0,\gamma_1} \quad 0 \le k \le 1,$$

where the constant C is independent of p and f.

Proof. It is immediate that (7.3) holds. Let $f(x) = x^n$ with $0 \le n \le p$ integer. Then

$$\begin{aligned} F(x,y) &= \frac{\sqrt{3}}{2y}\int_{x-\frac{y}{\sqrt{3}}}^{x+\frac{y}{\sqrt{3}}} t^n\,dt = \frac{\sqrt{3}}{2y(n+1)}\left[\left(x+\frac{y}{\sqrt{3}}\right)^{n+1} - \left(x-\frac{y}{\sqrt{3}}\right)^{n+1}\right] \\ &= \frac{\sqrt{3}}{2y(n+1)}\left[\left(x+\frac{y}{\sqrt{3}}\right) - \left(x-\frac{y}{\sqrt{3}}\right)\right] P_n(x,y) \\ &= \frac{1}{(n+1)} P_n(x,y) \in \mathcal{P}_p^1(T). \end{aligned}$$

Hence (7.2) holds.

To prove (7.4) we first extend f to a function defined on the entire x-axis $\Re$ so that [18]

$$\|f\|_{1/2,\Re} \le C\|f\|_{1/2,\gamma_1},$$

where we have used the same notation f to denote the extended function as well. Then by (7.1), $F_1(x,y)$ is well defined on the entire half plane $\Omega = \{(x,y) \mid y > 0\}$. For $(x,y) \in \Omega$ we have

(7.9) $$F_1(x,y) = \int_{-\infty}^{+\infty} f(t)H(x-t,y)\,dt = (f \star H(\cdot,y))(x)$$

where

(7.10) $$H(x,y) = \begin{cases} \frac{\sqrt{3}}{2y}, & -\frac{y}{\sqrt{3}} \le x \le \frac{y}{\sqrt{3}} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{g}(\xi)$ represent the Fourier transform of the function $g(x)$ in the x direction. Then by (7.9)

(7.11) $$\tilde{F}_1(\xi,y) = \tilde{f}(\xi)\tilde{H}(\xi,y)$$

where

(7.12) $$\tilde{H}(\xi,y) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{3}}{2y}\int_{-y/\sqrt{3}}^{y/\sqrt{3}} e^{i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\sin(\xi y/\sqrt{3})}{\xi y/\sqrt{3}}.$$

Let $\tilde{\Omega} = \{(\xi, y)|0 < y < 2\}$ and calculate the $H^1(\Omega)$ norm of $F_1(x,y)$. By Parseval's equality, we have using (7.11)

$$\|F_1\|^2_{H^1(\Omega)} = \|\tilde{F}_1\|^2_{H^1(\tilde{\Omega})} = \int_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\xi \tilde{H}(\xi,y)|^2 d\xi\ dy$$

$$+ \int_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\frac{\partial}{\partial y}\tilde{H}(\xi,y)|^2\, d\xi\ dy + \int_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\tilde{H}(\xi,y)|^2\, d\xi\ dy.$$

Now letting $z = y\xi/\sqrt{3}$ we obtain by (7.12),

$$\int_0^2 |\tilde{H}(\xi,y)|^2\ dy = \frac{1}{2\pi} \int_0^{2\xi/\sqrt{3}} \sqrt{3}\frac{\sin^2 z}{z^2}\frac{dz}{|\xi|} \leq \frac{C}{|\xi|+1}.$$

Hence

$$\int_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\xi \tilde{H}(\xi,y)|^2 d\xi\, dy \leq C \int_{-\infty}^{+\infty} |\xi|\, |\tilde{f}(\xi)|^2\, d\xi \leq C\|f\|^2_{1/2,\Re} \leq C\|f\|^2_{1/2,\gamma_1}. \tag{7.13}$$

Also

$$\frac{\partial}{\partial y}\tilde{H}(\xi,y) = \frac{\xi}{\sqrt{6\pi}}\left(\frac{\cos z}{z} - \frac{\sin z}{z^2}\right),$$

which is bounded at $z = 0$. Hence

$$\frac{1}{|\xi|}\int_0^{2\xi/\sqrt{3}} \left(\frac{\cos z}{z} - \frac{\sin z}{z^2}\right)^2 dz \leq \frac{C}{|\xi|+1},$$

and

$$\int_0^2 |\frac{\partial}{\partial y}\tilde{H}(\xi,y)|^2\ dy \leq C|\xi|,$$

so that

$$\int_{\tilde{\Omega}} |\tilde{f}(\xi)|^2 |\frac{\partial}{\partial y}\tilde{H}(\xi,y)|^2 d\xi\ dy \leq C\int_{-\infty}^{\infty} |\xi||\tilde{f}(\xi)|^2 d\xi \leq \|f\|^2_{1/2,\gamma_1}. \tag{7.14}$$

The third term can be bounded analogously. Using (7.13)–(7.14),(7.4) follows. Inequalities (7.7) and (7.8) follow immediately for $k = 0$, $k = 1$ and hence by an interpolation argument (see [6]), they hold for all $0 \leq k \leq 1$.

We prove now (7.5). Let the variable x be used to represent both the distance from A along γ_1 and the distance from A along γ_2. Denoting

$$G(x) = \frac{1}{x}\int_0^x f(t)\, dt \tag{7.15}$$

it is readily seen that

$$\|F_1^{[f]}\|_{k,\gamma_2^1} = \|G(x)\|_{k,I} \quad I = (0,1/2). \tag{7.16}$$

Using (9.9.1) of [12], p. 244 we obtain

$$\|G(x)\|_{0,I} \le C\|f\|_{0,\gamma_1^1}. \tag{7.17}$$

Further, integrating (7.15) by parts we have

$$G(x) = f(x) - \frac{1}{x}\int_0^x tf'(t)\,dt$$

and hence

$$G'(x) = f'(x) + \frac{1}{x^2}\int_0^x tf'(t)\,dt - f'(x) = -\frac{1}{x^2}\int_0^x (x-t)f'(t)\,dt + \frac{1}{x}\int_0^x f'(t)\,dt.$$

Using (9.9.5) of [12], p. 245 with $r = 2$ we obtain

$$\left\|\frac{1}{x^2}\int_0^x (x-t)f'(t)\,dt\right\|_{0,I} \le C\|f'\|_{0,I}$$

and by (9.9.1) of [12] p. 244 we obtain

$$\left\|\frac{1}{x}\int_0^x f'(t)\,dt\right\|_{0,I} \le C\|f'\|_{0,I}.$$

Hence

$$\|G'(x)\|_{0,I} \le C\|f'\|_{0,I}. \tag{7.18}$$

Combining (7.17) and (7.18) we obtain (7.5) for $k = 0$ and $k = 1$ and hence by the interpolation argument (7.5) holds for all $0 \le k \le 1$. The inequality (7.6) is essentially the same as (7.5) and Lemma 7.1 is completely proven. □

Let now $f = f_i \in \mathcal{P}_p(\gamma_i)$, $i = 1,2,3$. Then we denote by $F_i^{[f_i]}(x,y)$ the polynomial extension of f_i into T, defined for $i = 1$ by (7.1) and for $i = 2,3$ by (7.1) after properly rotating the coordinates. Obviously Lemma 7.1 is applicable for $i = 1,2,3$ when properly interpreted through the rotation of the coordinates.

Let γ_1 and γ_2 be the two sides of T (see Fig. 7.1) with the common vertex A. In the sequel, we will use norms of the form $\|\cdot\|_{k,\gamma_1\cup\gamma_2}$, defined by $\|u\|^2_{k,\gamma_1\cup\gamma_2} = \|u\|^2_{k,\gamma_1} + \|u\|^2_{k,\gamma_2}$ for $k < 1/2$ and for u continuous, $k > 1/2$ and

$$\|u\|^2_{1/2,\gamma_1\cup\gamma_2} \approx \|u\|^2_{1/2,\gamma_1} + \|u\|^2_{1/2,\gamma_2} + \int_0^1 \frac{(u_1(t) - u_2(t))^2}{t}\,dt$$

where u_i is the restriction of u to $\gamma_i, i = 1,2$ and t represents the distance from A along γ_1 or γ_2. These definitions may be extended to the case of different sides of T in an obvious way. The norm $\|\cdot\|_{1/2,\partial T}$ can be defined analogously.

We now prove

LEMMA 7.2. *Let T be the triangle as in Fig. 7.1 and f be continuous and such that $f_i = f\Big|_{\gamma_i} \in \mathcal{P}_p(\gamma_i)$, $i = 1,2$ where by $f\Big|_{\gamma_i}$ we denote the restriction on f on γ_i. Then there exists $\Phi_i \in \mathcal{P}_p(\gamma_i)$, $i = 1,2$ such that*

$$U = F_1^{[\Phi_1]} + F_2^{[\Phi_2]} \in \mathcal{P}_p^1(T), \tag{7.19}$$

$$U = f_i \quad \text{on} \quad \gamma_i, \quad i = 1,2, \tag{7.20}$$

$$\|U\|_{1,T} \le \|f\|_{1/2,\gamma_1\cup\gamma_2}, \tag{7.21}$$

$$\|\Phi_i\|_{k,\gamma_i} \le \|f\|_{k,\gamma_1\cup\gamma_2}, \quad i = 1,2, \quad 0 \le k \le 1, \tag{7.22}$$

$$\|\Phi_1\|_{k,\gamma_1^B} \le C\Big(\|f_1\|_{k,\gamma_1^B} + \sum_{j=1}^{2} \|f_j\|_{0,\gamma_j}\Big), \quad 0 \le k \le 1 \tag{7.23}$$

$$\|\Phi_2\|_{k,\gamma_2^C} \le C\Big(\|f_2\|_{k,\gamma_2^C} + \sum_{j=1}^{2} \|f_j\|_{0,\gamma_j}\Big), \quad 0 \le k \le 1, \tag{7.24}$$

where C is constant independent of p and f.

Proof. Let $\Phi_i \in \mathcal{P}_p(\gamma_i)$. Then as in Lemma 7.1 we define

$$G_i(x) = \frac{1}{x}\int_0^x \Phi_i(t)\,dt, \quad i = 1,2.$$

Condition (7.20) will be satisfied if

$$\begin{aligned} \Phi_i(x) + G_2(x) &= \Phi_1(x) + \frac{1}{x}\int_0^x \Phi_2(t)\,dt = f_1(x) \\ \Phi_2(x) + G_1(x) &= \Phi_2(x) + \frac{1}{x}\int_0^x \Phi_1(t)\,dt = f_2(x) \end{aligned} \tag{7.25}$$

hold for all $x \in I = (0,1)$. Since $f_i \in \mathcal{P}_p(I)$ it is easy to see that $\Phi_i \in \mathcal{P}_p(I)$ satisfying (7.25) exist. Φ_i are uniquely determined up to an additive constant $i = 1,2$. We now define

$$\begin{aligned} &\Psi_1(x) = \Phi_1(x) + \Phi_2(x), \quad \Psi_2(x) = \Phi_1(x) - \Phi_2(x), \\ &h_1(x) = f_1(x) + f_2(x), \qquad h_2(x) = f_1(x) - f_2(x) \end{aligned} \tag{7.26}$$

we see that $h_1 \in H^{1/2}(I), h_2 \in {}_AH^{1/2}(I)$, and ${}_A\|h_2\|_{1/2,I} \le \|f\|_{1/2,\gamma_1\cup\gamma_2}$, where we define

$${}_A\|h\|^2_{1/2,I} = \|h\|^2_{1/2,I} + \int_0^1 \frac{|h(t)|^2}{t}\,dt$$

and

$$ {}_A H^{1/2}(I) = \{u \in H^{1/2}(I) \mid {}_A\|u\|_{1/2,I} < +\infty\}. $$

Note that the space ${}_A H^{1/2}(I)$ is obtained by interpolation of $L^2(I)$ and the space

$$ {}_A H^1(I) = \{u \in H^1(I) \mid u(0) = 0\} $$

as ${}_A H^{1/2}(I) = [L^2(I), {}_A H^1(I)]_{1/2}$. By (7.25) we have

$$ \Psi_1(x) + \frac{1}{x}\int_0^x \Psi_1(t)\,dt = h_1(x), \tag{7.27} $$

$$ \Psi_2(x) - \frac{1}{x}\int_0^x \Psi_2(t)\,dt = h_2(x). \tag{7.28} $$

Here $\Psi_1(x)$ is unique, $\Psi_1(0) = \frac{h_1(0)}{2}$, while $\Psi_2(x)$ is unique up to the additive constant K. We first analyze (7.27). By differentiation we obtain

$$ \Psi_1{}' - \frac{1}{x^2}\int_0^x \Psi_1(t)\,dt + \frac{1}{x}\Psi_1 = h_1{}'. $$

Using (7.27) we obtain

$$ \Psi_1' + \frac{2\Psi_1}{x} = h_1' + \frac{h_1}{x}. \tag{7.29} $$

The homogeneous solution of (7.29) is $1/x^2$. A particular solution can be found by using the method of variation of constants. Hence, substituting $\Psi_1(x) = \frac{T(x)}{x^2}$ into (7.29) we obtain

$$ T'(x) = h_1' x^2 + h_1 x $$

from which

$$ \Psi_1(x) = \frac{1}{x^2}\int_0^x t^2 h_1(t)\,dt + \frac{1}{x^2}\int_0^x t h_1(t)\,dt. $$

Integrating by parts we obtain

$$ \Psi_1(x) = h_1(x) - \frac{1}{x^2}\int_0^x t h_1(t)\,dt. \tag{7.30} $$

the unique solution of (7.27). We show now that

$$ \|\Psi_1\|_{k,I} \le C\|h_1\|_{k,I}, \quad 0 \le k \le 1. \tag{7.31} $$

Let

$$F(x) = \int_0^x t h_1(t)\,dt = -\int_0^x (x-t)h_1(t)\,dt + x\int_0^x h_1(t)\,dt.$$

Then

$$-\frac{F(x)}{x^2} = \frac{G(x)}{x^2} - Q(x),$$

where

$$G(x) = \int_0^x (x-t)h_1(t)\,dt, \qquad Q(x) = \frac{1}{x}\int_0^x h_1(t)\,dt.$$

Using (9.9.4) of [12], p. 245 with $r = 2$ and (9.9.1) of [12],p. 244 we obtain

$$\|F(x)/x^2\|_{0,I} \leq \|G(x)/x^2\|_{0,I} + \|Q\|_{0,I} \leq C\|h_1\|_{0,I}$$

which yields (7.31) for $k = 0$. Next, differentiating (7.30) and integrating by parts we obtain

$$\Psi_1' = h_1' + \frac{2}{x^3}\int_0^x t h_1(t)\,dt - \frac{h_1}{x} = h_1' - \frac{1}{x^3}\int_0^x t^2 h_1'(t)\,dt.$$

Let

$$F(x) = \int_0^x t^2 h_1'(t)\,dt = \int_0^x (x-t)^2 h_1'(t)\,dt - x^2\int_0^x h_1'(t)\,dt + 2x\int_0^x t h_1'(t)\,dt.$$

We have then

$$\frac{F(x)}{x^3} = \frac{G(x)}{x^3} - Q(x) + R(x),$$

where

$$G(x) = \int_0^x (x-t)^2 h_1'(t)\,dt, \qquad Q(x) = \frac{1}{x}\int_0^x h_1'(t)\,dt$$

and

$$R(x) = \frac{2}{x^2}\int_0^x t h_1'(t)\,dt.$$

This gives

$$\|F(x)/x^3\|_{0,I} \leq \|G(x)/x^3\|_{0,I} + \|Q(x)\|_{0,I} + \|R(x)\|_{0,I}$$

The first two terms can be bounded once more by $C\|h_1'\|_{0,I}$ using (9.9.4) of [12], p. 245 with $r = 3$ and (9.9.1), p. 244. Moreover,

$$R(x) = \frac{2}{x^2}\left(-\int_0^x (x-t)h_1'(t)\,dt + x\int_0^x h_1'(t)\,dt\right)$$

so that $\|R\|_{0,I}$ can also be bounded by $\|h_1'\|_{0,I}$. This yields (7.31) for $k = 1$. By the interpolation argument (see [12]) we obtain immediately (7.31). Let us consider now (7.28). Differentiating it and using once more (7.28) we obtain

$$\Psi_2' = h_2' + \frac{h_2}{x}. \tag{7.32}$$

Integrating we obtain

$$\Psi_2(x) = h_2(x) - \int_x^1 \frac{h_2(t)}{t}\,dt, \tag{7.33}$$

which is the solution of (7.28) with $\Psi_2(1) = h_2(1)$. We wish to show now that

$$\|\Psi_2\|_{k,I} \le C_A\|h_2\|_{k,I}, \quad 0 \le k \le 1. \tag{7.34}$$

Using (7.33) and (9.9.9) from [12], p. 245 with $\alpha = 0$ we obtain

$$\|\Psi_2\|_{0,I} \le C\|h_2\|_{0,I}.$$

Since $h_2(0) = 0$, (7.32) yields

$$\Psi_2' = h_2' + \frac{1}{x}\int_0^x h_2'(t)\,dt$$

and by (9.9.1) of [12], p. 244 we obtain

$$\|\Psi_2\|_{1,I} \le C_A\|h_2\|_{1,I}.$$

An interpolation argument leads immediately to (7.34). Hence we have constructed solutions of (7.27),(7.28) such that (7.31) and (7.34) hold. We note that for $k = 1/2$, we have from (7.34) $\|\Psi_2\|_{1/2,I} \le C_A\|h_2\|_{1/2,I}$ and ${}_A\|h_2\|_{1/2,I}$ cannot be replaced by $\|h_2\|_{1/2,I}$. Coming back to (7.26), using $k = 1/2$ we see that for $i = 1, 2$,

$$\|\Phi_i\|_{1/2,\gamma_i} \le \|f\|_{1/2,\gamma_1\cup\gamma_2},$$

and applying Lemma 7.1 we obtain immediately (7.21) and also (7.22). Returning to (7.27), (7.28) we see that with $I^* = (1/2, 1)$,

$$\|\Psi_i\|_{k,I^*} \le C\Big(\|h_i\|_{k,I^*} + \|h_i\|_{0,I}\Big), \quad i = 1, 2.$$

Hence also

$$\|\Phi_i\|_{k,I^*} \le C\left(\|f_i\|_{k,I^*} + \sum_{j=1}^{2}\|f_j\|_{0,I}\right), \quad i = 1,2$$

which immediately leads to (7.23), (7.24). □

The following lemma is taken from [5].

LEMMA 7.3. *Let T be the triangle as before, f be continuous on ∂T, $f_2 = f_3 = 0$ and $f_1 \in \mathcal{P}_p(\gamma_1)$. Then there exists a polynomial $v \in \mathcal{P}_p^1(T)$ such that*

$$\|v\|_{1,T} \le C\|f_1\|_{1,\gamma_1}, \quad v = f_1 \quad \text{on } \gamma_1, \quad v = 0 \quad \text{on } \gamma_2, \gamma_3,$$

where C is a constant independent of f and p.

THEOREM 7.4. *Let T be the equilateral triangle shown in Fig. 7.1. f is continuous on ∂T and $f_i = f|_{\gamma_i} \in \mathcal{P}_p(\gamma_i)$, $i = 1,2,3$. Then there exists $U \in \mathcal{P}_p^1(t)$ such that $U = f$ on ∂T and*

$$\|U\|_{1,T} \le C\|f\|_{1/2,\partial T}$$

where the constant C is independent of p and f.

Proof. First we prove the theorem for the case when $f = 0$ on γ_2 and γ_3 and hence with $f_1 = f|_{\gamma_1}$ we have ${}_0\|f_1\|_{1/2,\gamma_1} \le \|f\|_{1/2,\partial T}$. By Lemma 7.2 we construct Φ_1, Φ_2 and $U_1 = F_1^{[\Phi_1]} + F_2^{[\Phi_2]}$. Then $U_1 \in \mathcal{P}_p(T)$, $U = f_1$ on γ_i, $i = 1,2$ and

$$\|U_1\|_{1,T} \le C\|f_1\|_{1/2,\gamma_1\cup\gamma_2} \le C\|f\|_{1/2,\partial T} \tag{7.35}$$

Denote by g_3 the trace of U_1 on γ_3. Then we have

$${}_0\|g_3\|_{1/2,\gamma_3} \le C\|f\|_{1/2,\partial T}$$

by (7.35) and the trace theorem. Because of (7.24), $\|\Phi_2\|_{1,\gamma_2^C} \le C\|f\|_{1/2,\partial T}$ and hence using Lemma 7.1 we have also

$$\|g_3\|_{1,\gamma_3^C} \le C\|f\|_{1/2,\partial T} \tag{7.36}$$

Let now analogously as before

$$U_2 = F_3^{[\Phi_3^{[1]}]} + F_1^{[\Phi_1^{[1]}]},$$

so that

$$U_2 \in \mathcal{P}_p^1(T),\ U_2 = g_3 \text{ on } \gamma_3,\ U_2 = 0 \text{ on } \quad \gamma_1$$

and

$$\|U_2\|_{1,T} \le C\,{}_0\|g_3\|_{1/2,\gamma_3} \le C\|f\|_{1/2,\partial T}. \tag{7.37}$$

Denote by $g_2^{[1]}$ the trace of U_2 on γ_2. Then $g_2^{[1]}(A) = g_2^{[1]}(C) = 0$. Because of (7.36), applying Lemma 7.1 and Lemma 7.2 analogously as before we conclude that

$$\|g_2^{[1]}\|_{1,\gamma_2} \le C\left(\|g_3\|_{1,\gamma_3^C} + \|g_3\|_{1/2,\gamma_3}\right) \le C\|f\|_{1/2,\partial T}.$$

Now applying Lemma 7.3, there is $U_3 \in \mathcal{P}_p^1(T)$ such that

$$\|U_3\|_{1,T} \le C\|g_2^{[1]}\|_{1,\gamma_2} \le C\|f\|_{1/2,\partial T} \tag{7.38}$$

and

$$U_3 = g_2^{[1]} \text{ on } \gamma_2,\ U_3 = 0 \text{ on } \gamma_1, \gamma_3.$$

Let now

$$V = U_1 - U_2 + U_3.$$

Then it is easy to see that $V \in \mathcal{P}_p^1(T)$, $V = f_1$ on γ_1, $V = 0$ on γ_2, γ_3 and because of (7.35), (7.37) and (7.38) we obtain

$$\|V\|_{1,T} \le C\|f\|_{1/2,\partial T}$$

which concludes the argument.

Secondly, now we will address the general case. By Lemma 7.2 we construct Φ_1, Φ_2 and $U_1 = F_1^{[\Phi_1]} + F_2^{[\Phi_2]}$. Then $U_1 \in \mathcal{P}_p^1(T)$, $U_1 = f_i$ on $\gamma_i, u = 1, 2$ and

$$\|U_1\|_{1,T} \le C\|f\|_{1/2,\gamma_1\cup\gamma_2} \le C\|f\|_{1/2,\partial T}.$$

Denote by g_3 the trace of U on γ_3 and $\tilde{g}_3 = g_3 - f_3$ on γ_3. Then

$$_0\|\tilde{g}_3\| \le C\|f\|_{1/2,\partial T}$$

and hence by the first part of the proof, there exists $V \in \mathcal{P}_p(T)$, $V = g_3$ on γ_3, $V = 0$ on γ_i, $i = 1, 2$, $\|V\|_{1,T} \le {}_0\|g_3\|_{1/2,\gamma_3}$. Hence taking $U = U_1 - V$ we obtain

$$\|U\|_{1,T} \le C\|f\|_{1/2,\partial T}$$

and $U = f$ on ∂T. This concludes the proof. □

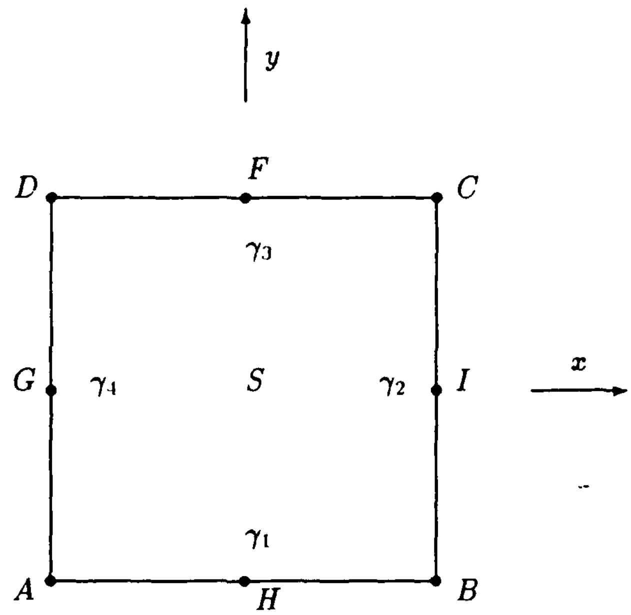
Let $S = \left\{(x, y) \mid |x| < 1, |y| < 1\right\}$ be a square and γ_i its sides as shown in Fig. 7.2.

THEOREM 7.5. *Let S be the square shown in Fig. 7.2 and f be continuous on ∂S and such that $f = f|\gamma_i \in \mathcal{P}_p(\gamma_i)$, $i = 1, \ldots, 4$. Then there exists $U \in \mathcal{P}_p^2(S)$ such that $U = f$ on ∂S and*

$$\|U\|_{1,S} \le C\|f\|_{1/2,\partial S}$$

where the constant C is independent of p and f.

FIG. 7.2. *Notation scheme for the square*



Proof. Let T be triangle shown in Fig. 7.1 and $Q = \{(\xi,\eta) | (\xi,\eta) \in T, \eta < \frac{3\sqrt{3}}{8}\}$ be the trapezoid shown in Fig 7.3. The mapping

$$(x,y) \mapsto (\xi,\eta), \quad \xi = \frac{1}{2} + \frac{3x}{16}(-y + \frac{5}{3}), \quad \eta = (1+y)\frac{3\sqrt{3}}{16}.$$

maps S onto Q. The mapping is obviously one-to-one, smooth, the Jacobian and its inverse are bounded, and the mapping is linear on ∂S. If $U \in \mathcal{P}_p^1(T)$ then

$$U(\xi,\eta) = \sum_{0 \le k+j \le p} a_{kj}\xi^k\eta^j = U(x,y) \in \mathcal{P}_p^2(S).$$

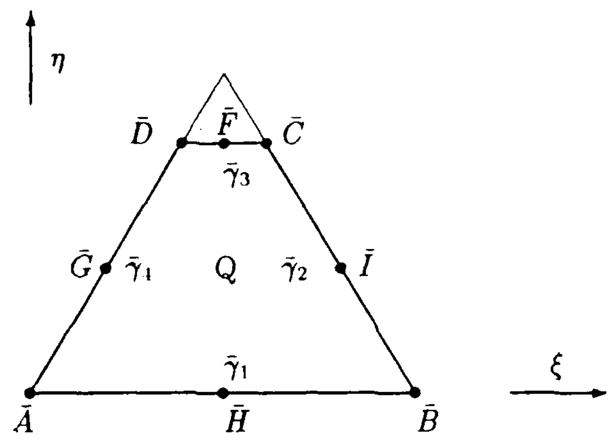
Because the used mapping is smooth, it preserves all norms under consideration. By $\bar{f}$ we denote the function on ∂Q obtained by transformation of f and by $\bar{f}_i$ we denote the restriction of $\bar{f}$ on $\bar{\gamma}_i$. First we construct the extension of $\bar{f}_1$ as in Lemma 7.1. Hence we can replace f by $g^{[1]}$ where $g^{[1]} = 0$ on one side of S and $\|g^{[1]}\|_{1/2,\partial S} \le C\|f\|_{1/2,\partial S}$. We can assume that we achieved, say, $g_1^{[1]} = 0$ which leads to the case $\bar{g}_1 = 0$. Extending $\bar{g}_1^{[1]}$ by zero we construct as in Lemma 7.2 the function $\tilde{v}_1 = g_1^{[1]}$ on $\tilde{\gamma}_1$, $\tilde{v}_1 = 0$ on $\tilde{\gamma}_4$ and

$$\|\tilde{v}_1\|_{1,T} \le \|f\|_{1/2,\partial S}.$$

Hence we can replace f by $g^{[2]}$, $g^{[2]} = 0$ on two neighboring sides of S say γ_2, γ_3 and $\|g^{[2]}\|_{1/2,\partial S} \le C\|f\|_{1/2,\partial S}$. Repeating once more the construction using Lemmas 7.2 and 7.1 we replace f by $g^{[3]}$ so that $g^{[3]} = 0$, say once more on γ_2 and γ_3, and

$$\|g^{[3]}\|_{1/2,\gamma_1\cup\gamma_4} \le C\|f\|_{1/2,\partial S}, \quad \|g^{[3]}\|_{1,\hat{A}B} \le C\|f\|_{1/2,\partial S}, \quad \|g^{[3]}\|_{1,\hat{A}G} \le C\|f\|_{1/2,S}.$$

FIG. 7.3. *Notation scheme for the trapezoid*



Repeating once more the process and using Lemmas 7.1 and 7.2 we reduce our original problem to the extension problem on S when $\|g_i^{[4]}\|_{1/2,\partial S} \leq C\|f\|_{1/2,\partial S}$ and $\|g_i^{[4]}\|_{1,\gamma_i} \leq C\|f\|_{1/2,\partial S}$. It is now easy to construct $v_2 \in \mathcal{P}_p^2(S)$ so that $v_2 = g^{[4]}$ on ∂S and

$$\|v_2\|_{1,S} \leq \sum_{i=1}^{4} \|g_i^{[4]}\|_{1,\gamma_i}$$

which leads immediately to the desired result. □

Theorem 7.5 is concerned with the space $\mathcal{P}_p^2(S)$. Theorem 7.5 does not hold for $\mathcal{P}_p^3(S)$, but there is a weaker statement easily available.

THEOREM 7.6. *Let the assumptions of Theorem 7.5 hold. Then there exists $U \in \mathcal{P}_p^3(S)$ such that $U = f$ on ∂S and*

$$\|U\|_{1,S} \leq Cp\|f\|_{1/2,\partial S}$$

Proof. By Schmidt's inequality [13], $\|f\|_{1,\gamma_i} \leq Cp^2\|f\|_{0,\gamma_i}$ and by interpolation,

$$\|f\|_{1,\gamma_i} \leq Cp\|f\|_{1/2,\partial S}.$$

Let us define

$$U_1 = f(x)\left(\frac{1-y}{2}\right)$$

and analogously U_i, $i = 2,3,4$. Then $V = \sum_{i=1}^{4} U_i$ is continuous on ∂S and $V - f$ is linear on every γ_i. Since $\|U_i\|_{1,S} \leq C\|f_i\|_{1,\gamma_i}$, we have

$$\|V\|_{1,S} \leq Cp\|f\|_{1/2,\partial S}.$$

Further with $V_i = V|_{\gamma_i}$ we also have

$$\|f - V_i\|_{1,\gamma_i} \leq Cp\|f\|_{1/2,\partial S}.$$

Let $\tilde{U}$ be the bilinear function such that $\tilde{U} = f - V$ on ∂S. Then also $\|\tilde{U}\| \leq Cp\|f\|_{1/2,\partial S}$, and $U = V + \tilde{U}$ is the desired extension. □

Now let us turn to extensions where only some boundary values are given.

THEOREM 7.7. *Let T be equilateral triangle, γ_1 one of its sides and v_1 the opposite vertex. Let $f_1 \in \mathcal{P}_p(\gamma_1)$. Then there exists $f \in \mathcal{P}_p^1(T)$ such that $f = f_1$ on $\gamma_1, f(v_1) = 0$, and*

$$\|f\|_{1,T} \leq C\|f_1\|_{1/2,\gamma_1}.$$

Proof. By Lemma 7.1, there is $F_1 \in \mathcal{P}_p^1(T)$ such that $F_1 = f_1$ on γ_1,

$$\|F_1\|_{1,T} \leq C\|f_1\|_{1/2,\gamma_1}, \qquad |F_1(v_1)| \leq C\|f_1\|_{1/2,\gamma_1}.$$

It suffices to put $F = f_1 - F_1(v_1)w$, where w is the linear function such that $w(v_1) = 1$ and $w = 0$ on γ_1. □

Here is a similar theorem for the square.

THEOREM 7.8. *Let S be square with sides γ_1 to γ_4 and $f_1 \in \mathcal{P}_p(\gamma_1)$. Then there exists $f \in \mathcal{P}_p^2(S)$ (resp. $\mathcal{P}_p^3(S)$) such that $f = f_1$ on γ_1, $f = 0$ on the opposite side γ_3, and*

$$\|f\|_{1,S} \leq C\|f_1\|_{1/2,\gamma_1}, \quad (\text{ resp. } \|f\|_{1,S} \leq Cp\|f_1\|_{1/2,\gamma_1})$$

with C independent on f and p.

Proof. As in the proof of Theorem 7.5, transform S to a trapezoid of Fig. 7.3, and use Lemma 7.1 to obtain function $g \in \mathcal{P}_p^2(S)$ such that $g = f_1$ on γ_1, and

$$\|g\|_{1,S} \leq C\|f_1\|_{1/2,\gamma}.$$

If follows immediately from the construction (7.1) that

$$\|g\|_{1,\gamma_3} \leq C\|f\|_{0,\gamma_1}.$$

Let S be as in Fig. 7.2, i.e., γ_1 characterized by $y = -1$, $x \in [-1,+1]$ and γ_3 by $y = 1$, $x \in [-1,+1]$. Let $v(x,y) = g(x)(y+1)$. Then $v = 0$ on γ_1, $v = g$ on γ_3, and

$$\|v\|_{1,S} \leq C\|g\|_{1,\gamma_3}.$$

Then $f = g - v$ is the desired extension and it holds that

$$\|g - v\|_{1,S} \leq \|g\|_{1,S} + \|v\|_{1,S} \leq C\|f_1\|_{1/2,\gamma_1}.$$

The second case follows from the first one using Theorem 7.6. □

Our last theorem is concerned with the extreme case of extending a function from a vertex onto a square.

THEOREM 7.9. *Let S be square with sides γ_1 to γ_1 and and A its vertex adjacent to γ_1 and γ_1. Then there exists a constant C such that for all $p \geq 1$, there is a function $v \in \mathcal{P}_p^3(S)$ such that $v(A) = 1$, $v = 0$ on the two sides γ_2 and γ_3 which are opposite to the vertex A, and*

$$\|v\|_{1,S} \leq C(1 + \log p)^{-1/2}.$$

Proof. It suffices to prove the theorem only for all sufficiently large p. Mapping the function from the second part of Theorem 6.2 onto the sides γ_1 and γ_1 so that the point -1 is mapped to the vertex A, and extending the function by reflection also on γ_2 and γ_3, we get a function u on ∂S such that

$$\|u\|_{1/2,\partial S} \leq C(1 + \log p)^{-1/2}, \qquad u(A) = 1.$$

Note that $\|u\|_{1/2,\partial S} = \sum_{i=1}^{4} \|u\|_{1/2,\gamma_i}$ (up to equivalence of norms) by symmetry. The function u is not a polynomial, but its restriction to each side is a polynomial function. By Theorem 7.5, there is an extension of u onto S such that $u \in \mathcal{P}_p^2$ and $\|u\|_{1,S} \leq C\|u\|_{1/2,\partial S}$. Let $w \in \mathcal{P}_1^3$ such that $w(A) = 1$ and $w = 0$ on γ_2 and γ_3, and put $v = wu$. Then

$$\|v\|_{1,S} \leq C\|u\|_{1,S} \leq C(1 + \log p)^{-1/2}, \qquad v(A) = 1,$$

and

$$v \in \mathcal{P}_{p+1}^2 \subset \mathcal{P}_{2(p+1)}^3.$$

□

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